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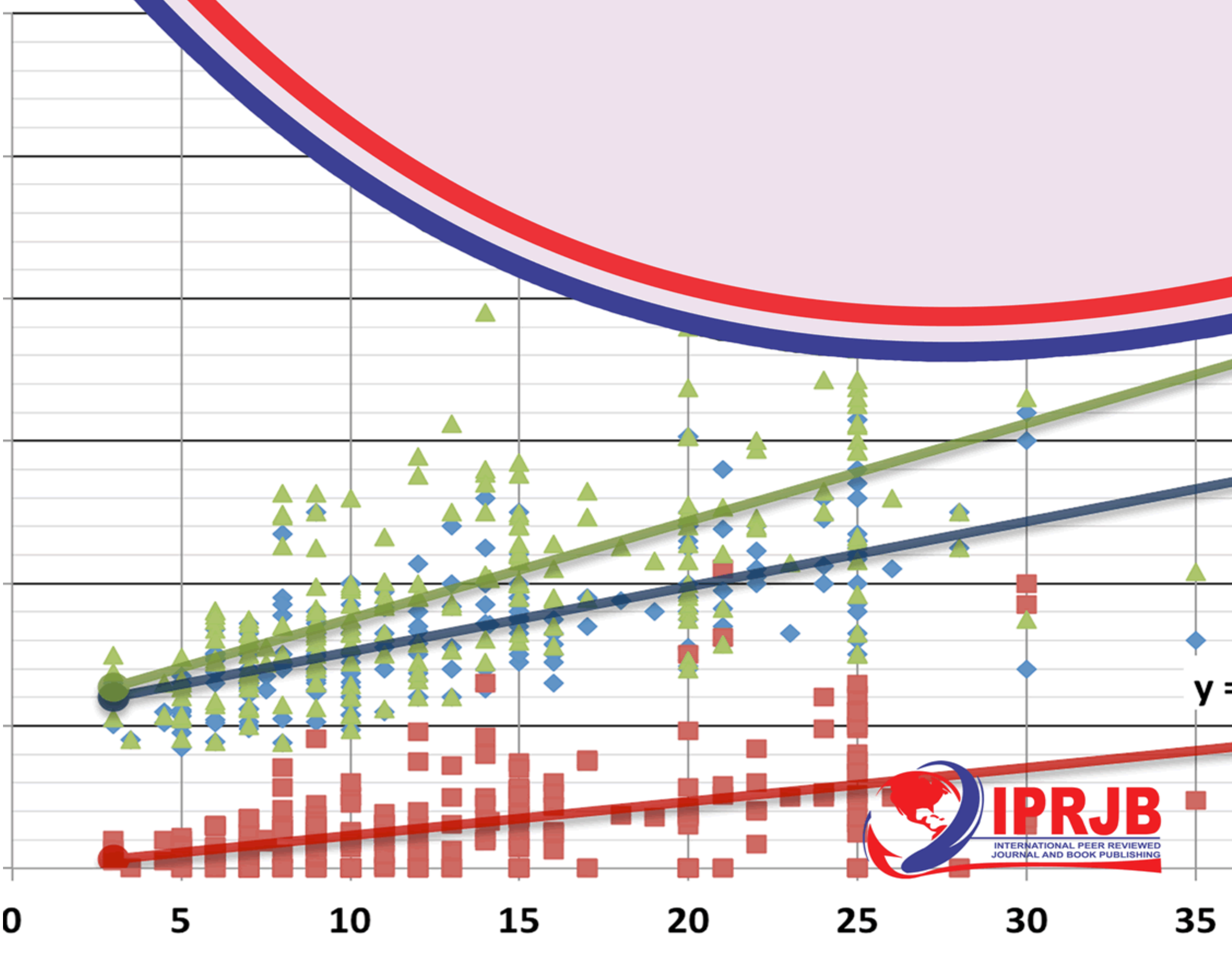

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SUM RULES FOR JACOBI MATRICES AND THEIR APPLICATIONS TO SPECTRAL THEORY

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Abstract

The study discusses the proof of and symmetric application of Cases sum rules for Jacobi matrices. Of special interest is a linear combination of these sum rules which have strictly positive terms. The complete classification of the spectral measure of all Jacobi matrices J for which J-J0 is Hilbert space –Achmidt. The study shows the bound of a Jacobi matrix. The description for the point and absolutely continuous spectrum, while for the singular continuous spectrum additional assumptions are needed**.** The study shows and prove a bound of a Jacobi matrix. And we give complete description for the point and absolutely continuous spectrum, while for the singular continuous spectrum additional assumptions are needed, we prove a characterization of a characteristic function of a row contraction operator and verify its defect operator. We also prove a commutability of an operator of this row contraction**.**

Keywords: *Sum Rules, Jacobi matrices, Spectral Theory*

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Section (1-1): Spectral Form for Jacobi Matrices:

The case of some rules and were efficiently used to relate properties of elements of a Jacobi matrix of certain class with its special properties. For instance spectral data of Jacobi matrices being a Hilbert space-Schmeidt perturbation of the free Jacobi matrix were characterization [42,101,135] and we suggest a modification of the method that permits us to work with higher order sum rules. We obtain sufficient conditions for a Jacobi matrix to satisfy certain constraints on its spectral measure. We consider a Jacobi matrix [129,124].

$$J = J(a,b) = \begin{bmatrix} b_0 & a_0 & 0 \\ a_0 & b_1 \cdots & \\ \vdots & \vdots & \ddots \end{bmatrix}$$

Where $a=\{a_k\}, a_{k>0}$ and $b=\{b_k\}, b_k \in \mathbb{R}$, We assume that J is a compact perturbation of the free Jacobi matrix J_0

$$J_0 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 \cdots & \\ \vdots & \vdots \ddots & \end{bmatrix} \quad (1)$$

A scalar spectral measure $\delta = \delta(J)$ is defined by the formula $((J-z)e_0, e_0) = \int_{\mathbb{R}} \frac{d\delta(x)}{x-z}$ with $z \in \mathbb{C} \backslash \mathbb{R}$, the absolutely continuous spectrum $\delta_{ac}(J)$ of J fills in [-2, 2]and the discrete spectrum consist of two sequences $\{x_j^{\pm}\}$with properties $\bar{x}_j < -2$, $\bar{x}_j \to 2$ and $x_j^+ > 2$, $x_j^+ \to 2$

Let $\partial_a = \{a_k - a_{k-1}\}$ for a given a and $k \in N$ we construct a sequence $\gamma_k(\alpha)$by formula $\gamma_k(a)_j = \alpha_j^k - \alpha_j ... \alpha_{j+k-1}$ where $\alpha = a-1$ and 1 is a sequence of units

Theorem (1-1-1) [87]:

Let $J = J(a,b)$ be a Jacobi matrix described above. If

(i) $a-1, b \in L^{m+1}, \partial_a, \partial_b \in L^2$

(ii) $\gamma_k(a) \in L', k = 3, [(m+1)/2]$ (2)

Then

$$(i') \int_{-2}^{2} \log \delta'(x).(4-x^2)^{m-\frac{1}{2}} dx > -\infty$$

$$(ii') \sum_j (x^{\pm 2} - 4)^{m+1/2} < \infty \quad (3)$$

When $m=1$ the theorem gives the fact of theorem (1-1-1)

Proof:

Define $\phi_m(J)$ as $\varphi_m(J) = \varphi_m(\delta) = \varphi_{m,1}(\delta) + \varphi_{m,2}(\delta)$

$$=\frac{1}{2\pi}\int_{-2}^{2}\log\frac{1}{\delta'(x)}.(4-x^2)^{m-\frac{1}{2}}dx+\sum_j G_m(x_j^{\pm}).$$

We have to show that $\phi_m(J)<\infty$. We put $a_N=\{(a_N)_k\}$ and $a'_N=\{(a'_N)_k\}$, where

$$(a_N)_k=\begin{cases}a_k, & k\le N,\\ 1, & k<N,\end{cases}\quad (a'_N)_k=\begin{cases}1, & k\le N,\\ a_k, & k<N,\end{cases}$$

Define sequences b_N , b'_N in the same way (of course, with $1's$ replaced by $0's$).

Let $J_N=J(a_N,b_N)$. As we readily see, $\dot{a}_N-1, b_N\to 0$, $\partial a'_N$, $\partial b'_N\to 0$, and $\gamma_k(a'_N)\to 0$ in corresponding norms, as $N\to\infty$ by the Lemma

(1-1-4) below, we have for $N'=N-m$

$$|\psi_m(J)-\psi_m(J_N)|\le\psi_m(a'_{N'},b_{N'})\le C_1(\|a'_{N'}-1\|_{m+1}+\|b_{N'}\|_{m+1}$$
$$+\|\delta a_{N'}\|_2+\|\delta b_{N'}\|_2+\sum_k\|\gamma_k(d_{N'})\|_1)$$

or $\psi_m(J_N)\to\psi_m(J), as\ N\to\infty$

on the other hand $(J_N-z)^{-1}\to(J-z)^{-1}$ $for\ z\in\square\setminus\mathbb{R}$, and consequently $\delta_N\to\delta$ weakly $\phi_{m,1}(\delta)\le\lim_N \operatorname{int}\phi_{m,1}(\delta_N)$ and $\lim\phi_{m,2}(\delta_N)=\phi_{m,2}(\delta)$ we bound the latter quantity $|\psi_{m,2}(J)|=\sum_j|G_m(x_j^{\pm})|\le C_2(\|a-1\|_{m+1}^{m+1}+\|b\|_{m+1}^{m+1})$ with some constant C_2. Summing up we obtain

$$\varphi(\delta)\le\lim_N\sup\varphi(\delta_N)=\lim_N\ \sup\psi(J_N)=\lim_{N\to\infty}\psi(J_N)=\psi(J)$$

The proof is complete. It is easy to give simple conditions sufficient for $\gamma_k(a)\in L'$ for the instance put

$(A_k(a))_j=\alpha_{j+1}+...+\alpha_{j+k-1}-(k-1)\alpha_j$, then relations $a-1eL^{m+1},\partial_a\in L^2$ and $A_k(a)\in L^{2(k,m)}$ $2(k,m)=(m+1)!(m+2-k)$ imply that $\gamma_k(a)\in L'$. In particular we have the following corollary.

Corollary (1-1-2) [87]:

Theorem (1-1-1) holds if conditions (i), (ii) are replaced with $A_k(a)\in L^{2(k,m)}$, $2(k,m)=(m+1)/(m+2-k)$, where $k=\delta,\left[\frac{m+1}{2}\right]$ we observe that relations (i) and (ii) are trivially true in the case of discrete Schrödinger operator i.e., when $J=J(1,b)$.

Corollary (1-1-3) [87]:

Then inequalities $(i')and(ii')$ let hold $J=J(1,b)$. If $b\in L^{m+1}$ **,** $\partial b\in L^2$**,** the corollary is still true if $b\in L^{m+2}$, m being even. The proof is a sum rule of a special type. First we obtain it assuming rank $(J-J_0)<\infty$. Applying methods we see that

$$\frac{1}{2\pi}\int_{-2}^{2}\log\frac{1}{\delta'(x)}.(4-x^2)^{m-\frac{1}{2}}dx+\sum_j G_m(x_j^{\pm})=\psi_m(J)$$

Where $\psi_m(J)=\psi_m(a,b)$ and $G_m(x)=(-1)^{m+1}C_0(x^2-4)^{m+\frac{1}{2}}+o(x^2-4)^{m+\frac{3}{2}}$ with $x\in R\setminus[-2,2]$, C_0 being a positive constant. where

$$\psi_m(J)=tr\left\{\sum_{k=1}^{m}\frac{(-1)^{k+1}}{2^{2k+1}k}\left(J^{2k}-J_0^{2k}\right)-\frac{(2m-1)!!}{(2m)!!}\log A\right\} \quad (4)$$

Where $A=diag\{a_k\}$ and $\tilde{C}_m^k=\frac{m!!}{(m-k)!!k!!}$. Notation $k!!$ is used for "even" or "odd" factorials.

Lemma (1-1-4) [5]:

Let $J=J(a,b)$ we have

$$|\psi_m(J)|\le C_1\left(\|a-1\|_{m+1}+\|b\|_{m+1}+\|\partial_a\|_2+\|\partial b\|_2+\sum_{k=3}^{[(m+1)/2]}\|\gamma_k(a)\|_1\right) \quad (5)$$

Where C_1 depends on T only. Above, norms $\|.\|_p$ refer to the standard L^p-space norms. We begin with considering expressions $\mathrm{tr}\left(J^{2k}-J_0^{2k}\right)$ arising in (4). Defining $V=J-J_0=J(a-1,b)$ we have

$$tr\left(J^{2k}-J_0^{2k}\right)=tr\sum_{p=1}^{2k}\sum_{i_1+\ldots+i_p=2k-p}VJ_0^{i_1}\ldots VJ_0^{i_p}$$

We prove the lemma in steps.

Proof:

First we bounded summands corresponding to $P=\frac{m+1}{2},m$ in [87]. We get $\left|tr\left(V^pF_p(J_0)\right)\right|\le\left\|V^pF_p(J_0)\right\|_{S_1}\le\left\|F_p(J_0)\right\|,\left\|V^p\right\|_{S_1}$ and for these $P^{'s}$

$$\left\|V^p\right\|_{S_1}\le C_{10}\left\|V^{m+1}\right\|_{S_1}\le C_{10}\left(\|a-1\|_{m+1}^{m+1}+\|b\|_{m+1}^{m+1}\right) \quad (6)$$

With the constant depending on $\|V\|$. Similarly $|tr\,\alpha^p|\le C_{11}\|a-1\|_{m+1}^{m+1}$, let $p=3,m$ now. As we already mentioned in [134]

$$V^p=\sum_{j=0}^{p}\left(S^ip_{p,j}(a,b)+p_{p,j}(a,b)\bar{S}^j\right).$$

It is easy to show by induction that the polynomials $P_{p,p}(a,b)$ are particularly simple. Namely $P_{p,p}(a,b)=\alpha\alpha_{(1)}\ldots\alpha_{(p-1)}$ yields that

$$trV^pF_p(J_0)=(-1)^p\frac{(2m-1)!!}{2p(2m)!!}trV^pJ_{0,p}$$

$$=(-1)^p\frac{(2m-1)!!}{2p(2m)!!}tr\left(P_{p,p}(a,p)+P_{p,p}(a,b)_p\right)$$

$$=(-1)^p\frac{(2m-1)!!}{2p(2m)!!}\sum_j\alpha_j\alpha_{j+1}\ldots\alpha_{j+(p-1)}$$

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Since $trV^pJ_{0,s} = 0$ for $s \geq p+1$. Hence $tr\left(V^pF_p(J_0)+(-1)^{p+1}\frac{(2m+1)!!}{p(2m)!!}\alpha^p\right)$

$$= (-1)^{p+1}\frac{(2m-1)!!}{2p(2m)!!}\sum_j \alpha_j^p - \alpha_j\alpha_{j+1}...\alpha_{j+(p-1)} \quad \text{and we obtain that}$$

$$\left|trV^pF_p(J_0)+(-1)^{p+1}\frac{(2m-1)!!}{2p(2m)!!}\alpha^p\right| \leq C_{12}\left\|\gamma_p(a)\right\|_1 \qquad (7)$$

Where C_{12} depends on p, m and sequences $\gamma_k(a)$ are defined in [134]

Observe that $\gamma_p(a)=0$ when $p=1$. Furthermore we have for $p=2$ that

$$\sum_j\left(\alpha_j^2-\alpha_j\alpha_{j+1}\right)=\frac{1}{2}\sum_j\left(\alpha_j^2-2\alpha_j\alpha_{j+1}+\alpha_{j+1}^2\right)$$

$$=\frac{1}{2}\sum_j\left(\alpha_j-\alpha_{j+1}\right)^2=\frac{1}{2}\|\partial a\|_2^2$$

So the left hand-side of (7) for $p=2$ can be estimated by $C_{13}\|\partial a\|_2^2$. It is also clear that inclusion $\alpha \in L^{m+1}$ and $\partial a \in L^2$ give that $\gamma_p(a) \in L'$ for $p > m/2+1$. Indeed we have

$$\alpha^p - \alpha\alpha_{(1)}...\alpha_{p-1} = \sum_{k=1}^{p}\alpha^{p-k}\left(\alpha-\alpha_{p-k}\right)\alpha_{(p-(k-1))}...\alpha_{p-1}$$

The terms in the latter sum look like $\alpha(i_1)...\alpha_{(2p-1)}\left(\alpha-\alpha_{(i_p)}\right)$ for some $i=(i_1,...,i_p)$. Obviously $\alpha-\alpha_{ip}=a-a_{ip}=\partial\, a \in L^2$. Applying the Holder inequality $\sum_k a_k...a_{p+k} \leq \sum_k\left(\sum_{j=1}^{p}\frac{1}{q_j}a_{j,k}^{q_j}\right)$ with $a_{j,k}=\left|\left(\alpha_{i_j}\right)_k\right|, q_j=2(p-1)$ for $j=1, p-1$ and $a_{p,k}=\left|\left(\alpha-\alpha_{i_p}\right)_k\right|, q_p=\frac{1}{2}$ we get that

$$\left\|\alpha^p-\alpha\alpha_{(1)}...\alpha_{(p-1)}\right\|_1 \leq C_{14}\|\partial a\|_2^2+\|\alpha\|_{2(p-1)}^{2(p-1)}$$

Which is finite for $p > m2+1$. Thus gathering the above argument which is complete(see [134])we complete the proof of the lemma

Lemma (1-1-5) [87]:

Let $i=(i_1,...,i_p)$ and $\sum_s i_s = n$ then

$$VJ_0^{i_1}...VJ_0^{i_p} = V^pJ_0^n + \sum_{\substack{L_1+L_2+L_3=p \\ p_1+p_2+p_3=n}} C_{1,p}J_0^{p_1}V^{L_1}\left[V^{L_2},J_0^{p_2}\right]V^{L_3}J_0^{p_3}$$

$$+\sum_i^{m,p_i}A_k\left[V,J_0\right]B_k\left[V,J_0\right]C_k$$

Where $p=(p_1,p_2,p_3), 1=(L_1,L_2,L_3)$ and A_k, B_k, C_k are some bounded operators

Lemma (1-1-6) [87]

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Let $\sum_s i = 2k - p$ we have $\left| tr\left(VJ_0^{i_1}...VJ_0^{i_p} - V^p J_0^{k-p}\right)\right| \le C_3\left(\|\partial a\|_2 + \|\partial b\|_2\right)$

With C_3 depending on $\|V\|$ only. The lemma exactly bounded ,we may assume that operators V and J_0 to commute we estimating $\psi_m(J)$

$$\psi'_m(J) = tr\left\{\sum_{p=1}^{2m} V^p F_p(J_0) - \frac{(2m-1)!!}{(2m)!!}\log(I+\tilde{\alpha})\right\} \qquad (8)$$

Where $\tilde{\alpha} = diag\{a_k\} = A - I$ and

$$F^p(J_0) = \sum_{k=[(p+1)/2]}^{m} \frac{(-1)^{k+1}}{2^{2k+1}} \tilde{C}_{2m-1}^{2k-1} C_{2k}^p J_0^{2k-p}$$

Here C_k^p is a usual binomial coefficient, observe that for $p \ge m+1$ we have

$$\left| tr\left(V^p F_p(J_0)\right)\right| \le \|F_p(J_0)\| \left\|V^p\right\|_{\delta_1} \le C_4\left(\|a-1\|_{m+1}^{m+1} + \|b\|_{m+1}^{m+1}\right)$$

Where $\|.\|_{\delta_1}$ is the norm in the class of nuclear operators, hence it remains to bound the first m terms in (8) we have

$$\log(1+\tilde{\alpha}) = \sum_{p=1}^{2m} \frac{(-1)^{p+1}}{p}\tilde{\alpha} + o\left(\tilde{\alpha}^{2m+1}\right)$$

Set $J_{0,p}$ to be a symmetric matrix with 1's on p-th auxiliary diagonals and o's elsewhere the following lemma holds.

Lemma (1-1-7) [87]:

We have $F_p(J_0) = (-1)^{p+1}\frac{(2m-1)!!}{2p(2m)!!} J_{0,p}$

Combining this with explicit form of V^p and the series expansion for $\log(I+\tilde{\alpha})$ we get the required bound (7).

Section (1-2): Spectral Properties of Self-adjoint Extensions

Let A be a closed symmetric operator on a separable Hilbert space h. If A has equal deficiency indices $n_\pm(A)$ =dim$\left(h \ominus ran(A \pm iI)\right)$, then A has a lot of self-adjoint extensions. These self-adjoint extensions can be labeled by the so-called Weyl function $M(.)$ [82, 83, and 84]. The generalization is based on concept of a boundary triple $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for A^* being an abstract generalization of the Green's identity. Here $\mathcal{H}$ is a separable Hilbert space with dim $(\mathcal{H}) = n_\pm(A)$ and Γ_0 and Γ_1 are linear mapping from dom(A^*) to $\mathcal{H}$ so that Green's identity is satisfied [108,119].

The problem is the following. Let $M(.)$ be the Weyl function of a certain self-adjoint extensions A_0 of A, introducing the associated scalar Weyl function $M_h(.) = (M(.)h, h)$, $h \in \mathcal{H}$ is it possible to localize the different spectral subsets of A_0 knowing the boundary values

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$M_h(x+i0), x \in \mathbb{R}$ of the associated scalar Weyl function. Let $\mathcal{H}$ be separable Hilbert space. Recall that an operator function $F(.)$ with values in $[\mathcal{H}]$ is said to be a Hirglotz or Nevanlina function or R-function if holomorphic in $\mathbb{C}_+$ and for every $z \in \mathbb{C}_+$ the operator $F(z)$ in $\mathcal{H}$ is dissipative i.e., $Sm(F(z)) = \dfrac{\left(F(z)-F(z)^*\right)}{2i} \geq 0$. In the following we prefer the notion R-function. The class of R–functions with values in $[\mathcal{H}]$ is denoted by $R_{\mathcal{H}}$ .If $F(.) \in (R_{\mathcal{H}})$ then there exist bounded self-adjoint operator L in K, a bounded non-negative operator $R \geq 0$ with $R|K \ominus \mathcal{H} = 0$ such that

$$F(z) = C_0 + C_1 z + R^{\frac{1}{2}}(I_k + zL)(L-z)^{-1} R^{\frac{1}{2}}|\mathcal{H}, \quad z \in \mathbb{C}_+ \qquad (9)$$

Denoting by $E_L(.)$ the spectral measure of the self-adjoint operator L one immediately obtains from (9) the representation

$$F(z) = C_0 + C_1 z + \int_{-\infty}^{\infty}\left(\frac{1}{t-z} - \frac{t}{1+t^2}\right) d\sum_F(t), \ z \in \mathbb{C}_+ \qquad (10)$$

Where $\sum_F(.)$ is an operator valued Borel measure on $\mathfrak{R}$ given by

$$d\sum_F(t) = (1-t^2) R^{\frac{1}{2}} dE_L(.) R^{\frac{1}{2}}, \ t \in \mathbb{R} \qquad (11)$$

the measure $\sum_F(.)$ is self-adjoint and obeys

$$\int_{-\infty}^{+\infty} \frac{1}{1+t^2} d\sum_F(t) \in [\mathcal{H}] \qquad (12)$$

In contrast to spectral measures of self-adjoint operators it is not necessary true that ran $\sum \delta_1$ is orthogonal to ran$\left(\sum \delta_2\right)$ for adjoint Borel sets δ_1 and δ_2.

However the measure $\sum_F(.)$ is uniquely determined by the R-function $F(.)$.

The integral in (10) is understood in the strong sense in the following $\sum_F(.)$ is called the spectral measure of $F(.)$ defined by

$$\sum_F(t) = \begin{cases} \sum_F(0,t) : t > 0 \\ 0 \qquad\quad : t = 0 \\ -\sum_F(t,0) : t < 0 \end{cases} \qquad (13)$$

The distribution function $\sum_F(.)$ is strongly left continuous and satisfies the condition

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$$\sum_F(t)=\sum_F(t)^*,\ \sum_F(s)\le\sum_F(t),-\infty<s<t<\infty$$

The distribution function $\sum_F(.)$ is called the spectral function of $F(.)$.

We note that the spectral function $\sum_F(.)$ can be obtained by the Stieltjes transformation:

$$\frac{1}{2}\sum_F(t+0)+\sum_F(t)-\frac{1}{2}\sum_F(s+0)+\sum(s)=w-\lim_{y\to 0}\frac{1}{\pi}\int_s^t Sm\left(F\left(x+iy\right)\right)dx\ ,\quad t.s\in\square \qquad (14)$$

Where it is used that the spectral function is strongly left continuous.

A will always denote a closed symmetric operator with equal deficiency indices $n_+(A)=n_-(A)$ [97,140,147,148].

We can assume that A is simple. This means that A has no self-adjoint parts. **Definition (1-2-1) [96]:**

A triple $\Pi=\{\mathcal{H},\Gamma_0,\Gamma_1\}$ consisting of an auxiliary Hilbert space $\mathcal{H}$ and linear mapping $\Gamma_i:dom\left(A^*\right)\to\mathcal{H},\ i=0,1$ is called a boundary triple for the adjoint operator $A^*\to\mathcal{H}$, $i=0,1$ is called a boundary triple for the adjoint operator A^* of A if

(i) The second Green's formula takes place

$$\left(A^*,f\right)-\left(f,A^*g\right)=\left(\Gamma_1 f,\Gamma_0 g\right)-\left(\Gamma_0 f,\Gamma_1 g\right),\quad f,g\in dom\left(A^*\right) \qquad (15)$$

(ii) The mapping $\Gamma=\{\Gamma_0,\Gamma_1\}:dom\left(A^*\right)\to\mathcal{H}\oplus\mathcal{H}$ is surjective

Definition (1-2-2) [96]:

(i) A closed linear relation θ in $\mathcal{H}$ is closed subspace θ of $\mathcal{H}\oplus\mathcal{H}$.

(ii) The closed linear relation θ is symmetric if $\left(g_1,f_2\right)-\left(f_1,g_2\right)=0$ for all $\{f_1,g_1\},\{f_2,g_2\}\in\theta$

(iii) The closed linear relation θ is self-adjoint if it is maximal symmetric.

Definition (1-2-3) [96]:

Let $\{\mathcal{H},\Gamma_0,\Gamma_1\}$ be a boundary triple for A^*

(i) for every self-adjoint relation θ in $\mathcal{H}$ we put

$$D^\theta\left\{f\in dom\left(A^*\right):\Gamma_0 f,\Gamma_1 f\in\theta\right\},A^\theta=A^*\big|D^\theta \qquad (16)$$

(ii) In particular we set $A_i=A^{\theta_i},i=0,1,\ if\theta_i,i=0,1$

(iii) If $\theta=G(B)$ where B is an operator on $\mathcal{H}$, then we set $A^B A^\theta$

Proposition (1-2-4) [96]:

Let $\{\mathcal{H},\Gamma_0,\Gamma_1\}$ be a boundary triple for A^* then for every self-adjoint relation θ in $\mathcal{H}$ the operator A^θ given by definition (1-2-3) is self-adjoint extension of A the mapping $\theta\mapsto A^\theta$ from the set of self-adjoint extensions in $\mathcal{H}$ onto the set Ext_A of self-adjoint extensions of A is bijective. It is well known that Weyl function are an important tool in the direct and inverse spectral theory of singular Sturm-Liouville operators.

Definition (1-2-5) [96]:

Let $\{\mathcal{H},\Gamma_0,\Gamma_1\}$ be a boundary triple for the operator A^*. The Weyl function of A corresponding to the boundary triple $\{\mathcal{H},\Gamma_0,\Gamma_1\}$ is the unique mapping

$M(.):\rho(A_0)\to\mathcal{H}$ satisfying

$$\Gamma_1 f_z = M(z)\Gamma_0 f_z, \quad f_z\in N_z, \quad z\in\rho(A_0) \qquad (17)$$

Where $N_z=\ker(A^*-zI)$ above implicit definition of the Weyl function is correct and the Weyl function $M(.)$ is a R-function obeying

$$o\in\rho\left(Sm\left(M\left(i\right)\right)\right)$$

Definition (1-2-6)[96]:

A closed linear relation θ in $\mathcal{H}$ is called boundedly invertible if the inverse relation $\theta^{-1}=\{g,f\}\in\mathcal{H}\times\mathcal{H}:\{f,g\in\theta\}$ is the graph of a bounded operator defined on $\mathcal{H}$. we say $\lambda\in$ belong to the resolvent set $\rho(\theta)$ if the closed linear relation $\theta-\lambda T=\{\{f,g-\lambda f\}:\{f,g\}\in\theta\}$ is boundedly invertible.

Proposition (1-2-7) [96]:

Let A be a simple closed densely defined symmetric operator in h. Suppose that $\{\mathcal{H},\Gamma_0,\Gamma_1\}$ is a boundary triple for A^* $M(.)$ is the corresponding Weyl function ,θ a self-adjoint relation in $\mathcal{H}$ and $\lambda\in\rho(A_0)$. Then the following holds.

(i) $\lambda\in\rho(A^\theta)$ if and only if $0\in\rho(\theta-M(\lambda))$.

(ii) $\lambda\in\delta_\tau\left(A^\theta\right)$ if and only if $0\in\delta_\tau\left(\theta-M\left(\lambda\right)\right),\tau=p,c$

If A is a simple symmetric operator then the Weyl function $M(.)$ determines the pair $\{A,A_0\}$ up to unitary equivalence. We shall often say that $M(.)$ is the Weyl function of the pair $\{A,A_0\}$ or simply of A_0. We can prove $M_1(.)$ and $M_2(.)$ with values in $[\mathcal{H}_1]$ and $[\mathcal{H}_2]$ are connected via

$$M_2(z)=K^*M_1(z)K+D \qquad (18)$$

Where $D=D^*\in[\mathcal{H}_2]$ and $K\in[\mathcal{H}_2,\mathcal{H}_1]$ is boundedly invertible. With each boundary triple we can associate a so-called γ-field $\gamma(.)$ corresponding to π is defined by

$$\gamma(z)=\left(\Gamma_0|N_z\right)^{-1}:\mathcal{H}\to N_2, z\in\rho(A_0) \qquad (19)$$

One can easily check that

$$\gamma(z)=(A_0-z_0)(A_0-z)^{-1}\gamma(z_0), z.z_0\in\rho(A_0) \qquad (20)$$

And consequently $\gamma(.)$ is a γ-field. The γ-field and the Weyl function $M(.)$ are related by

$$M(z)-M(z_0)^*=(z-\bar{z}_0)\gamma(z_0)^*\gamma(z), \qquad z,z_0\in\rho(A_0) \qquad (21)$$

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The relation (21) means the $M(.)$ is a θ_2 -function of a pair $\{A, A_0\}$.Further we note that if A is simple then $N_z, z \in \rho(A_0)$ is generating with respect to A_0 too .

Let μ be a Borel measure on $\square$. A support of μ is a set S such that $\mu(\square \setminus S)=0$ we note that $S \subseteq \tilde{S}$ implies that $\tilde{S}$ is a support too. Measures μ and v on $\Re$ are called orthogonal if some of their supports are disjoint. The topological support $S(\mu)$ of μ is the smallest closed set which is a support of μ .According to the Lebesgue-Jordan\decomposition $\mu = \mu_s + \mu_{ac}, \mu_s = \mu_{pp} + \mu_{sc}$. Where $\mu_s, \mu_{pp}, \mu_{sc}$ and are the corresponding singular pure point, singular continuous and absolutely continuous measures of μ respectively. We set

$S_T(\mu) = S(\mu_T), \quad T = s, pp, sc, ac$ the set $S_s(\mu), S_{pp}(\mu), S_{sc}(\mu), S_{ac}(\mu)$ are closed and called singular, pure point, singular continuous and absolutely continuous supports of μ, we denote that the closed supports $S_s(\mu), S_{pp}(\mu), S_{ac}(\mu)$ and $S_{sc}(\mu)$ are not generally mutually disjoint to obtain mutually disjoint supports we introduce the following sets .

$$S_0'(\mu) = \{t \in \square : d\mu(t)dt \text{ existsand } d\mu(t) = \infty\} \quad (22)$$

$$S_{pp}'(\mu) = \{t \in \square : \mu(\{t\}) \neq 0\} \quad (23)$$

$$S_{sc}'(\mu) = \left\{t \in \square : d\mu(t)d(t) \text{ exists } \frac{d\mu(t)}{dt} = \infty \text{ and } \mu(t) = 0\right\} \quad (24)$$

$$S_{ac}'(\mu) = \left\{t \in \square : \frac{d\mu(t)}{dt} \text{ exists and } 0 < d\mu(t)/dt < \infty\right\} \quad (25)$$

Where the distribution function $\mu(.)$ is similar to (13) defined by it turns out that. Since the sets $S_T'(\mu)$,$T = s, pp, sc$ are of Lebesgue measure zero and mutually disjoint we find that for any Borel set $\chi \subseteq \Re$ one has

$$\mu(\chi \cap S_T'(\mu)) = \mu_T(\chi), T = s, pp, sc, ac \quad (26)$$

The sets $S_s'(\mu), S_{pp}'(\mu), S_{sc}'(\mu)$, and $S_{ac}'(\mu)$ singular pure point ,singular continuous and absolutely continuous supports of μ respectively. We note that

$$S_{pp}(\mu) = \overline{S_{pp}'(\mu)} \text{and } S_\tau(\mu) \subseteq \overline{S_\tau'(\mu)} \subseteq S(\mu), \ \tau = s, sc, ac \quad (27)$$

In general it is not possible to replace inclusion by equalities, let now $\sum(.)$ be a measure with values in $\{\mathcal{H}\}$ the measure $\sum(.)$ admit a Lebesque- Jordan decomposition $\sum = \sum^{ac}, \sum^{s}, \sum^{pp} + \sum^{sc}$ .As above the notation

$$S_s \sum = S\sum^{s}, S_{pp} \sum = S\sum^{pp}, S_{sc} \sum = S\left(\sum^{sc}\right) \text{and } S_{ac}\left(\sum\right) = S\left(\sum_{ac}\right)$$

stand for the singular pure point , singular continuous and absolutely. We get

$$S_p\left(\sum\right) = \{\tau \in \square\} : \sum(\{\tau\}) \neq 0 \quad (28)$$

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we have $S_p\left(\sum\right)=S_{pp}\left(\sum\right)$ and $\overline{S_p\left(\sum\right)}=S_{pp}\left(\sum\right)$ with each operator-valued measure $\sum(.)$ we can associate a scalar measure $\sum_h(.)=\left(\sum(.)h,h\right)$, $h\in\mathcal{H}$. In the following we are interested in the problem whether the spectral properties of the operator valued measure $\sum(.)$ can be characterized by a family of scalar measures. To this end let $\tau=\{h\}_{k=1}^{N}$, $1\le N\le+\infty$ be a total set in $\mathcal{H}$ with we associate the family $\sum_{h_k}\{.\}_{k=1}^{N}$, of scalar measures. Let us introduce the following sets.

$$S_s'\left(\sum_;\tau\right)=\cup_{k=1}^{N}S_s'\left(\sum_{h_k}\right) \quad (29)$$

$$S_{pp}'\left(\sum_;\tau\right)=\cup_{k=1}^{N}S_{pp}'\left(\sum_{h_k}\right) \quad (30)$$

$$S_{sc}'\left(\sum_;\tau\right)=\cup_{k=1}^{N}S_{sc}'\left(\sum_{h_k}\right)|\,S_{pp}'\left(\sum\right) \quad (31)$$

$$S_{ac}'\left(\sum_;\tau\right)=\cup_{k=1}^{N}S_{ac}'\left(\sum_{h_k}\right)|\,S_s'\left(\sum\right) \quad (32)$$

Lemma (1-2-8) [96]:

Let $\mathcal{H}$ be a separable Hilbert space and $T=\{h_k\}_{k=1}^{N}$, $1\le N\le+\infty$ be a total set in $\mathcal{H}$.Then the sets $S_s'\left(\sum_;\tau\right), S_{pp}'\left(\sum_;\tau\right), S_{sc}'\left(\sum_;\tau\right)$ and $S_{ac}'\left(\sum_;\tau\right)$ are singular ,pure point ,singular continuous and absolutely continuous supports of $\sum(.)$ respectively i.e.,

$$\sum\left(\chi\cap S_\tau'\left(\sum_;\tau\right)\right)=\sum^{\tau}(\chi),\ \tau=s,pp,sc,ac \quad (33)$$

For any Borel set $\chi\subseteq\Re$. In particular the following relations hold.

$$S_p'\sum=S_{pp}'\left(\sum_;\tau\right) \text{ and}$$

$$S_p\left(\sum\right)\subseteq\overline{S_p'\left(\sum_;\tau\right)}\subseteq S\left(\sum\right),\ \tau=s,sc,ac \quad (34)$$

Proof:

By the Lebesgue-Jordan decomposition one easily gets that for each $h\in\mathcal{H}$ We have

$$\left(\sum^{\tau}(\chi)h,h\right)=\sum_{h,\tau}(\chi),\ \tau=s,pp,sc,ac \quad (35)$$

For any Borel set $\chi\in\Re$ where $\sum_{h,T}(.)$arises from the Lebesgue-Jordan decomposition of the scalar measure $\sum_h(.)$. Let $\tau=s$. Since mes $S_s'\left(\sum_;\tau\right)=0$

We get

$$\left(\sum\left(\chi\cap S_s'\left(\sum_;\tau\right)\right)h_k,h_k\right)=\sum_{h_k}\left(\chi\cap S_s'\left(\sum_;\tau\right)\right)=\sum_{h_{k,s}}\left(\chi\cap S_s'\left(\sum_;\tau\right)\right) \quad (36)$$

For any $h_k\in\tau$ using (35),(36) and

$$\sum_{h_{k,s}}\left(\chi\cap S_s'\left(\sum_;\tau\right)\right)=\sum_{h_{k,s}}\left(\chi\cap S_s'\left(\sum_;\tau\right)\cap S_s'\sum_{h_k}\right)$$

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$$=\sum_{h_{k,s}}\left(\chi\cap S_s'\left(\sum_{;}\tau\right)\right)=\sum_{h_{k,s}}(\chi) \tag{37}$$

We find $\left(\sum\left(\chi\cap S_s'\left(\sum_{;}\tau\right)\right)h_k,h_k\right)=\left(\sum^{s}(\chi)h_k,h_k\right)$ for any $h_k\in\tau$. Since τ is total we finally obtain $\sum\left(\chi\cap S_s'\left(\sum_{;}\tau\right)\right)=\sum^{s}(\chi)$ for any Borel set $\chi\in\mathbb{R}$. Similarly we prove the statements for $T=pp,sc,ac.$

Let $x\in S_{pp}'\sum_{;}\tau$. Then there is $h_k\in\tau$ such that $x\in S_{pp}'\left(\sum_{h_k}\right)$. Hence $\sum_{h_k}(\{x\})=\left(\sum(\{x\})h_k,h_k\right)\neq 0$ which yields $\sum(\{x\})\neq 0$ or $x\in S_p\left(\sum\right)$, i.e. $S_{pp}'\left(\sum_{;}\tau\right)\subseteq S_p\left(\sum\right)$ conversely if $x\in S_p\left(\sum\right)$ then there is a $h\in\mathcal{H}$ such that $\sum_h(\{x\})\neq 0$. If this is not the case then for each $h_k\in T$ one has $\sum_h(\{x\})=\left(\sum(\{x\})h_k,h_k\right)=0$. Since T is total this yields $\left(\sum(\{x\})h_k,h_k\right)=\sum_h(\{x\})=0$

For each $h\in\mathcal{H}$ Contrary to the assumption. however, if there is a $h_k\in T$ such that \` $\sum_{h_k}(\{x\})\neq 0$ then $x\in S_{pp}'\left(\sum_{;}\tau\right)$, i,e. $S_p\left(\sum\right)\subseteq S_{pp}'\sum_{;}\tau$ hence $S_p\left(\sum\right)=S_{pp}'\sum_{;}\tau$. Further from (33) we get

$$S_T\left(\sum\right)\subseteq\overline{S_T'\left(\sum_{;}\tau\right)},\ \tau=s,pp,sc,ac$$

Taking (27) into account we get $S_T'\left(\sum_{h_k}\right)\subseteq S\left(\sum_{h_k}\right)$, $\tau=s,sc,ac,sc$ for each $h_k\in\tau$. Since $S\left(\sum_{h_k}\right)\subseteq S\left(\sum\right)$ for each $h_k\in T$ we get $S_T'\left(\sum_{;}\tau\right)$ which immediately proves (34). Taking (20) and (21) into account we obtain that $C_1=0$ which leads to the representation.

$$M(z)=C_0+\int_{\mathbb{R}}\left(\frac{1}{t-z}-\frac{t}{1+t^2}\right)d\sum(t),\ z\in\mathbb{C} \tag{38}$$

Lemma (1-2-9) [96]:

Let A be a simple densely defined closed symmetric operator on the a separable Hilbert space with $n_+(A)=n_-(A)$. Further, let $\Pi=[\mathcal{H},\Gamma_0,\Gamma_1]$be a boundary triple of A^* with Weyl function $\sum(.)$. If $E_{A_0}(.)$ is the spectral measure of $A_0=A^*|\ker(\Gamma_0)(\in Ext_A)$ and $\sum(.)$ that of the integral representation (38) of the Weyl function $\sum(.)$. Then the measure $E_{A_0}(.)$ and $\sum(.)$ are equivalent. In particular one has $\delta_\tau(A_0)=\delta_\tau\left(\sum\right)$.

Theorem (1-2-10) [96]:

Let A be a simple densely defined closed symmetric operator on a separable Hilbert space h with $n_+(A)=n_-(A)$. Further let $\Pi=[\mathcal{H},\Gamma_0,\Gamma_1]$be a boundary triple of $A^*$with Weyl function $M(.)$.

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If $E_{A_0}(.)$ is the spectral measure of $A_0 = A^* \ker \Gamma_0 (\in Ext_A)$ and $\sum(.)$ that of the integral representation (38) of the Weyl function $M(.)$, then for each total set

$\tau = \{h_k\}_{k=1}^{N}$, $1 \le N \le +\infty$ in $\mathcal{H}$ the sets $S'_s\left(\sum_; \tau\right), S'_{pp}\left(\sum_; \tau\right), S'_{sc}\left(\sum_; \tau\right)$ and $S'_{ac}\left(\sum_; \tau\right)$.

Singular pure point, singular continuous, and absolutely continuous supports of $E_{A_0}(.)$ respectively, i.e. we have

$$E_{A_0}\left(\chi \cap S'_T\left(\sum_; \tau\right)\right) = E^T_{A_0}(\chi) \tag{39}$$

For each Borel set $\chi \in \Re$. In particular the relations $\delta_p(A_0) = S'_{pp} \sum_; \tau$ and

$\delta_\tau(A_0) \subseteq S'_\tau \sum_; \tau \subseteq \delta(A_0)$, $\tau = s, sc, ac$ hold.

Proof:

Since by lemma (1-2-8) the sets $S'_T(\delta;\tau)$, $\tau = s, pp, sc, ac$ are supports of $\sum(.)$, one immediately gets from lemma (1-2-9) that the same sets are supports of $E_{A_0}(.)$ of the same type, i.e., (39) holds . If $x \in S'_{pp}\left(\sum_; \tau\right)$ then there is at least one $k = 1,2,\ldots, N$ such that

$$\left(E_{A_0}(\{x\})\,\gamma(i)\,h_k, \gamma(i)\,h_k\right) \ne 0$$

Hence $x \in \delta_p(A_0)$ conversely, if $x \in \delta_p(A_0)$ then due to the fact that $\gamma(i)\tau$ is Generating for $E_{A_0}(.)$ then is at least one $k = 1,2,\ldots., N$ such that

$$\left(E_{A_0}(\{x\})\,\gamma(i)\,h_k, \gamma(i)\,h_k\right) \ne 0$$

Hence $x \in S'_{pp} \sum_; \tau$ which proves $\delta_p(A_0) = S'_{pp}\left(\sum_; \tau\right)$ the relations $\delta_s(A_0) \subseteq \overline{S'_\tau\left(\sum_; \tau\right)} \subseteq \delta(A_0)$, $\tau = s, sc, ac$ are consequences of lemma (1-2-8) and lemma (1-2-9) .we characterize the spectral properties of the operator-valued measure $\sum(.)$ using the boundary behavior of the Weyl-function $M(.)$. A first step is to develop a corresponding theory for scalar measure μ which satisfies

$$\int_{\Re} \frac{d\mu(t)}{1+t^2} < +\infty \tag{40}$$

Let us associate with μ the Poisson integral

$$V(z) = \int_{\Re} \frac{y\,d\mu(t)}{(t-x)^2 + y^2}, z = x + iy \in \mathbb{C}_+ \tag{41}$$

Which defines a positive harmonic function in $\mathbb{C}_+$. Conversely it is well known that each positive harmonic function $V_1(z)$ in $\mathbb{C}_+$ admits the representation $V_1(z) = ay + V(z)$ with $a \ge 0$ and $V(z)$ of the form (40) and (41). Below we summarize some well-known facts on positive harmonic function

Proposition (1-2-11) [96]:

Let μ be a positive Radan measure obeying (40) and let $V(z)$be a positive harmonic function in $z = x + iy \in \mathbb{C}_+$ defined by (41). Then one has.

(i) for any $x \in \mathbb{R}$ the $\lim V(x+io) = \lim V(x+iy)$ exists and is finite , if and only if symmetric derivative $D_\mu(x)$

$$D_\mu(x) = \lim_{\varepsilon \to 0} \frac{\mu(x+\varepsilon) - \mu(x-\varepsilon)}{2\varepsilon} \qquad (42)$$

Exists and is finite. In this case one has

$$V(x+io) = \pi D_\mu(x) \qquad (43)$$

(ii) if the symmetric derivative $D_\mu(x)$ exists and is infinite the $V(z) \to +\infty$ as $z \to\!> x$

(iii) for each $x \in \Re$ one has $Sm(z-x)V(z) \to \mu(\{x\})$ as $z \to\!> x$

(iv) $V(z)$ converges to a finite constant as $z \to\!> x$, if and only if the derivative $d\mu(t)dt$ exists at $t = x$ and is finite.

The symbol $\to\!>$ means that the limit $\lim_{r\to 0} V(x + re^{i\theta}), x \in \Re$ exists uniformly in $\theta \in [\varepsilon, \pi - \varepsilon]$ for each $\varepsilon \in (0, \pi/2)$. Proposition (1-2-11) allows us to introduce measures satisfying (40) the following sets $z = (x + iy)$

$$S''_s(\mu) = \{x \in \mathbb{R} : \mathrm{V}(z) \to \infty\} \text{ as } z \to\!> x \qquad (44)$$

$$S''_{pp}(\mu) = x \in \mathbb{R} : \mathrm{Sm} \lim_{z \to> x} (z - x) V(z) > 0 \qquad (45)$$

$$S''_{sc}(\mu) = \{x \in \mathbb{R} : \mathrm{V}(z) \to \infty \textit{ and } (z-x)V(z) \to 0 \textit{ as } z \to\!> x\} \qquad (46)$$

$$S''_{sc}(\mu) = \{x \in \mathbb{R} : \mathrm{V}(x+i0) \textit{exists and } 0 < V(x+i0) < \infty\} \qquad (47)$$

Obviously the sets $S''_s(\mu)$and $S''_{ac}(\mu)$as well as $S''_{pp}(\mu)$, $S''_{sc}(\mu)$, and $S''_{ac}(\mu)$are mutually disjoint. By proposition (1-2-15) one immediately gets that $S'_{pp}(\mu) = S''_{pp}(\mu)$ and

$$S'_\tau(\mu) \subseteq S''_\tau(\mu) \subseteq S(\mu) \qquad (48)$$

Indeed the relation $S'_{pp}(\mu) = S''_{pp}(\mu)$is a consequence of (iii).By (ii) we get

$$S'_s(\mu) \subseteq S''_s(\mu)$$

Similarly we prove $S'_{sc}(\mu) \subseteq S''_{sc}(\mu)$ using (ii) and(iii).Finally the relation $S'_{ac}(\mu) \subseteq S''_{ac}(\mu)$ follows from (i). We note that it can happen that $S'_{sc}(\mu) \neq 0$ and the inclusion $S'_{sc}(\mu) \subseteq S''_{sc}(\mu)$ is strict even if $\mu_{sc} = 0$. Furthermore we note that from (26) and the inclusion $S'_T(\mu) \subseteq S''_T(\mu)$, $\tau = s, pp, sc, ac$ we find that

$$\mu(\chi \cap S''_\tau(\mu)) = \mu_\tau(x) \qquad (49)$$

For any Borel set $\chi \in \mathbb{R}$. Now we are going to characterize the spectral parts of the extension A_0 by means of boundary values of the Weyl function $M(.)$.

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Using the integral representation (38) of the Weyl function we easily get that

$$V_h(z)=\int_{\mathbb{R}}\frac{y}{(x-t)^2+y^2}\,d\sum_h(t)=Sm\big(M_h(z)\big),z\in\mathbb{C},\quad h\in\mathcal{H} \qquad (50)$$

Where $M_h(z)=\big(M(z)h,h\big),z\in\mathbb{C},\quad h\in\mathcal{H}$ (51)

The function $M_h(.)$ is a scalar R-function. Since $M_h(.)$ arises from the Weyl function we call it the associated scalar Weyl function $V_h(.)$ is imaginary

part of the associated scalar Weyl function $M_h(.)$ and the theory developed we can relate the boundary behavior at the real axis the imaginary part of associated scalar Weyl functions with the spectral properties of the self-adjoint extension A_0. To this end in addition to (29) and (32) we introduce.

$$S_s''\left(\sum\nolimits_{;}\tau\right)=\cup_{k=1}^{N}S_s''\left(\sum\nolimits_{h_k}\right) \qquad (52)$$

$$S_{pp}''\left(\sum\nolimits_{;}\tau\right)=\cup_{k=1}^{N}S_{pp}''\left(\sum\nolimits_{h_k}\right) \qquad (53)$$

$$S_{sc}''\left(\sum\nolimits_{;}\tau\right)=\cup_{k=1}^{N}S_{sc}''\left(\sum\nolimits_{h_k}\right)\setminus S_{pp}''\left(\sum\right) \qquad (54)$$

$$S_{ac}''\left(\sum\nolimits_{;}\tau\right)=\cup_{k=1}^{N}S_{ac}''\left(\sum\nolimits_{h_k}\right)\setminus S_s''\left(\sum\right) \qquad (55)$$

By definition the sets $S_s''\left(\sum_{;}\tau\right)$ are disjoint. They holds for $S_{pp}''\left(\sum_{;}\tau\right)$. Furthermore we denote that the sets $S_T''\left(\sum_{;}\tau\right)$ have Lebesgue zero, i.e., mes $S_T''\left(\sum_{;}\tau\right)=0$, $\tau=s,pp,sc.$, it turns out that the sets $S_T''\left(\sum_{;}\tau\right)$ in theorem (1-2-14) can be replaced by the sets $S_T''\left(\sum_{;}\tau\right)$

Theorem (1-2-12) [96]:

Let A be a simple densely defined closed symmetric operator on a separable Hilbert space H with $n_+(A)=n_-(A)$. Further, let $\Pi=\{\mathcal{H},\Gamma_0,\Gamma_1\}$ be a boundary triple of A^* with Weyl function $M(.)$. If $E_{A_0}(.)$ is the spectral measure of $A_0=A^*|\ker\Gamma_0(\in Ext_A)$ and total set $T=\{h_k\}_{k=1}^{N}$, $1\le N\le+\infty$ in $\mathcal{H}$ the sets $S_s''\left(\sum_{;}\tau\right)$, $S_{pp}''\left(\sum_{;}\tau\right)$, $S_{sc}''\left(\sum_{;}\tau\right)$ and $S_{as}''\left(\sum_{;}\tau\right)$ are singular ,pure point , singular continuous and absolutely continuous supports of $E_{A_0}(.)$ respectively , i.e., we have

$$E_{A_0}\left(\chi\cap S_\tau''\left(\sum\nolimits_{;}\tau\right)\right)=E_{A_0}^{\tau}(\chi),\tau=s,pp,sc,ac \qquad (56)$$

For each Borel set $\chi\in\mathbb{R}$. In particular it hold $\delta_p(A_0)=S_{pp}''\left(\sum_{;}\tau\right)$ and

$$\delta_\tau(A_0)\subseteq S_\tau''\left(\sum\nolimits_{;}\tau\right)\subseteq\delta(A_0),\tau=s,sc,ac.$$

Proposition (1-2-13) [96]:

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Let $\phi(.)$ be a scalar R-function. Then for almost all $x \in$ the limit $\phi(x+i0) = \lim_{y \to 0} \phi(x+i0)$ exists and moreover in this case one has $\varphi(x+io) = \lim_{z \to > x} \varphi(z)$.

Theorem (1-2-14) [96]:

Let A be a simple densely defined closed symmetric operator on a separable Hilbert space h with $n_+(A) = n_-(A)$ Further let $\Pi = \{\mathcal{H}\Gamma_0, \Gamma_1\}$ be a boundary triple of A^* with Weyl function $M(.)$ and let $E_{A_0}(0)$ be the spectral measure of the self- adjoint extension A^* of A . If $\tau = \{h_k\}_{k=1}^{N}$,$1 \le N \le +\infty$ is a total set in $\mathcal{H}$ then sets $\Omega_s(M;\tau)$, $\Omega_{pp}(M;\tau)$, $\Omega_{sc}(M;\tau)$ and $\Omega_{ac}(M;\tau)$ are supports of $E_{A_0}(.)$ respectively. i.e., we have

$$E_{A_0}\left(\chi \cap \Omega_\tau(M;\tau)\right) = E_{A_0}^{\tau}(\chi), \tau = s, pp,, sc.ac \quad (57)$$

For each Borel set $\chi \in \Re$ In particular it holds $\delta_p(A_0) = \Omega_{pp}(M;\tau)$ and $\delta_\tau(A_0) \subseteq \overline{\Omega_\tau(M;\tau)} \subseteq \delta_\tau(A_0)$ for $\tau = s, sc, ac$. We note that the inclusions $\delta_s(A_0) \subseteq \overline{\Omega_s(M;\tau)}$ and $\delta_{sc}(A_0) \subseteq \overline{\Omega_{sc}(M;\tau)}$ of theorem (1-2-14) may be strict even if $\delta_{sc}(A_0)$ is empty.

Let $\mu(.)$ be a Borel measure on $\Re$ and let $\chi \subseteq$ be a Borel set the set

$$CL_{ac}(x) = x \subseteq \; : \text{mes}\left((x - \varepsilon, x + \varepsilon) \cap x\right) > 0 \; \forall \varepsilon > \delta \quad (58)$$

is called the absolutely continuous closure of set x obviously the set $CL_{ac}(x) \in \bar{x}$ is always closed and one has

Lemma (1-2-15) [96]:

Let $\phi(.)$ be a scalar R-function which has the representation (10) then $S_{ac}(\mu) = CL_{ac}(\Omega_{ac}(\phi))$

Proof:

If $x \notin= CL_{ac}(\Omega_{ac}(\phi))$ then there is an $\in > 0$ such that $mes(x-\varepsilon, x+\varepsilon) \cap \Omega_{ac}(\varphi) = \theta$

$$\mu_{ac}(x-\varepsilon, x+\varepsilon) = \mu_{ac}(x-\varepsilon, x+\varepsilon) \cap \Omega_{ac}(\varphi) = 0 \quad (59)$$

Hence $x \notin S(\mu_{ac}) = S_{ac}(\mu)$ which yields $S_{ac}(\mu) \subseteq CL_{ac}(\Omega_{ac}(\phi))$ conversely if $x \notin S_{ac}(\mu)$ then there is an $\varepsilon > 0$ such that $(x-\varepsilon, x+\varepsilon) \cap \Omega_{ac}(\mu) = 0$ then $\mu_{ac}(x-\varepsilon, x+\varepsilon) = 0$ using

$$\mu_{ac}(x-\varepsilon, x+\varepsilon) = \mu_{ac}(x-\varepsilon, x+\varepsilon) \cap \Omega_{ac}(\phi) \int_{(x-\varepsilon,x+\varepsilon)\cap\Omega_{ac}(\phi)} \frac{d\mu(t)}{dt} dt = 0 \quad (60)$$

and proposition (1-2-11) (i) and (vi) one gets

$$\mu_{ac}(x-\varepsilon, x+\varepsilon) = \frac{1}{\pi} \int_{(x-\varepsilon,x+\varepsilon)\cap\Omega_{ac}(\phi)} Sm(\phi(\tau + i0)) d\tau = 0 \quad (61)$$

Hence $Sm(\phi(t+i0))dt = 0$ for $a.e.t \varepsilon (x-\varepsilon, x+\varepsilon) \cap \Omega_{ac}(\phi)$. However by definition of the set $\Omega_{ac}(\phi)$ one has $Sm(\phi(\tau + i0))dt > 0$ for all $\tau \varepsilon \Omega_{ac}(\phi)$ which implies $mes((x-\varepsilon, x+\varepsilon) \cap \Omega_{ac}(\phi)) = 0$

Hence $x \notin CL_{ac}(\Omega_{ac}(\phi))$ or equivalent $CL_{ac}(\Omega_{ac}(\phi)) \subseteq S_{sc}(\mu)$.

Proposition (1-2-16) [96]:

Let A be a simple densely defined closed symmetric operator a separable Hilbert space with $n_+(A) = n_-(A)$. Further let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triple of A^* with Weyl function $M(.)$ If $\tau = \{h_k\}_{k=1}^{N}$,$1 \le N \le +\infty$ is a total set in $\mathcal{H}$ then the absolutely continuous spectrum of the self-adjoint extension A_0 of A is given by.

$$\delta_{ac}(A_0) = \overline{\cup_{k=1}^{N} CL_{ac}(\Omega_{ac}(M_{h_k}))} \quad (62)$$

Theorem (1-2-17)[96]:

Let A be a simple densely defined closed symmetric operator on a separable Hilbert space h with $n_+(A) = n_-(A)$. Further, let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triple of A^* with Weyl function $M(.)$.

If $\tau = \{h_k\}_{k=1}^{N}$,$1 \le N \le +\infty$ is a total set in $\mathcal{H}$, then for the self-adjoint extension A_0 of A the following conclusions are valid :

(i) The self-adjoint extension A_0 of A has no point spectrum within the interval (a,b) . i.e., $\delta_{pp}(A_0) \cap (a,b) = \theta$ if and only if for each $k = 1,2,\dots N$ one has

$$\lim_{y \to 0} y M_{hk}(x + iy) = 0 \quad (63)$$

for all $x \in (a,b)$. In this case the following relation holds

$$\delta(A_0) \cap (a,b) = \frac{\delta_c(A_0) \cap (a,b)}{\cup_{k=1}^{N} \Omega_{ac}(M_{hk}) \cup} = \overline{\cup_{k=1}^{N} \Omega_{ac}(M_{hk})} \cap (a,b) \quad (64)$$

(ii) The self-adjoint extension A_0 of A has no singular continuous spectrum within the interval (a,b), $i.e. \delta_{ac}(A_0) \cap (a,b) = \theta$ if for each $k = 1,2,\dots N$ the set $\Omega_{ac}(M_{hk}) \cap (a,b)$ is countable in particular, if $(a,b) | \Omega_{ac}(M_{hk})$ is countable.

(iii) The self-adjoint extension A_0 of A has no absolutely continuous spectrum within the interval (a,b) .i.e., $\delta_{ac}(A_0) \cap (a,b) = \theta$ if and only if for each $k = 1,2,\dots N$ the condition

$$Sm(M_{hk}(x + i0)) = 0 \quad (65)$$

holds for a.e. $x \in (a,b)$. in this case we have

$$\delta_s(A_0) \cap (a,b) = \overline{\Omega_s(M;\tau)} \cap (a,b)$$

Proof:

(i) If condition (65) is satisfied for all $x \in (a,b)$ and all $k = 1,2,\dots N$, then a simple computation shows that $\lim_{z \to > x} (z - x) M_{hk} = 0$ holds for all $x \in (a,b)$ and each $k = 1,2,\dots N$ too. Therefore $\Omega_{pp}(M_{hk}) \cap (a,b) = 0$ for $k = 1,2,\dots N$ whichyields $\Omega_{pp}(M;T) \cap (a,b) = 0$ theorem (1-2-14). Implies $\delta_p(A_0) \cap (a,b) = 0$ which yields $\delta_{pp}(A_0) \cap (a,b) = 0$.

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(ii) Conversely if $\delta_{pp}(A_0)\cap(a,b)=0$ then $\delta_p(A_0)\cap(a,b)=0$ again by theorem (1-2-14) we find $\delta_{pp}(A_0)\cap(a,b)=0$ therefore $\delta_{pp}(A_0)\cap(a,b)=0$ for each $k=1,2,...N$. However this implies that $\lim_{z\to>x}(z-x)M_{hk}(z)=0$ which yields $\lim_{y\to 0} yM_{hk}(x+iy)=0$ for all $x\in[a,b]$ and each $k=1,2,...N$. The first of relation (64) is consequence of $\delta(A_0)=\delta_{pp}(A_0)U\delta_c(A_0)$ and $\delta_{pp}(A_0)\cap\ (a,b)=0$. The second part of relation (64) is a consequence of theorem (1-2-18) which shows that

$$\delta_\tau(A_0)\subseteq\overline{\Omega_\tau(M;\tau)}=\overline{\cup_{k=1}^{N}\Omega_\tau(M;\tau)}\subseteq\delta(A_0),\ \tau=sc,ac \qquad (67)$$

and $\delta_c(A_0)=\delta_{sc}(A_0)U\delta_{ac}(A_0)$.Both facts imply that $\delta_c(A_0)\cap(a,b)\subseteq$

$$\overline{\cup_{k=1}^{N}\Omega_{ac}(M_{hk})}\cup\overline{{}_{k=1}^{N}\Omega_{ac}(M_{hk})}\cap(a,b)\subseteq\delta(A_0)\cap(a,b)=\delta_c(A_0)\cap(a,b) \qquad (68)$$

Which proves (64)

(ii) By (53) we gets that $S'_{ac}\left(\sum_{hk}\right)=S'\left(\sum_{hk,sc}\right)\subseteq S''_{sc}\left(\sum_{hk}\right)\Omega sc(M_{hk})$. Therefore if $\Omega_{ac}(M_{hk})\cap(a,b)$ is countable, then so is $S''_{ac}\left(\sum_{hk}\right)\cap(a,b)$ this yields that the singular continuous measure $\sum_{hh,sc}(.)$ is supported within the interval (a,b) on a countable set. However this implies that $\sum_{hk,sc}(a,b)=0$ for each $k=1,2,...,N$ and every $h\in\mathcal{H}$ one has $\sum_{hK,sc}(a,b)=0$ which yields $\sum^{sc}(a,b)=0$. Therefore by lemma(1-2-9) one gets $E_{A_0}^{sc}(a,b)=0$ which proves $\delta_{sc}(A_0)\cap(a,b)=0$. If $(a,b)\setminus\Omega_{ac}M_{hk}$ is countable, then by $\Omega_{sc}(M_{hk})\subseteq(a,b)\setminus\Omega_{ac}(M_{hk})$ the set $\Omega_{sc}(M_{hk})$ is countable too which completes the proof (ii).

(iii) If for each $k=1,2,...,N$ the condition (65) holds for a.e. $x\in(a,b)$ each $\varepsilon>0$ one has $mes(x-\varepsilon,x+\varepsilon)\cap\Omega_{ac}(M_{hk})\cap(a,b)=\theta$ hence $CL_{ac}(\Omega_{ac}(M_{hk}))\cap(a,b)=0$ taking proposition (1-2-16) into account we find $\delta_{ac}(A_0)\cap(a,b)=0$. Conversely if $\delta_{ac}(A_0)\cap(a,b)=0$ then proposition (1-2-16) for each $k=1,2,...,N$ we have $CL_{ac}(\Omega_{ac}(M_{hk}))\cap(a,b)=CL_{ac}(\Omega_{ac}(M_{hk}))\cap(a,b)=0$ Which verifies condition (65) for a.e $x\in(a,b)$. Using $\delta(A_0)\cap(a,b)=\delta_s(A_0)$ and

$\delta_s(A_0)\subseteq\overline{\Omega_s(M;\tau)}\subseteq\delta(A_0)$ which was proved in theorem (1-2-14)

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