Journal of
Statistics and Acturial Research
(JSAR)

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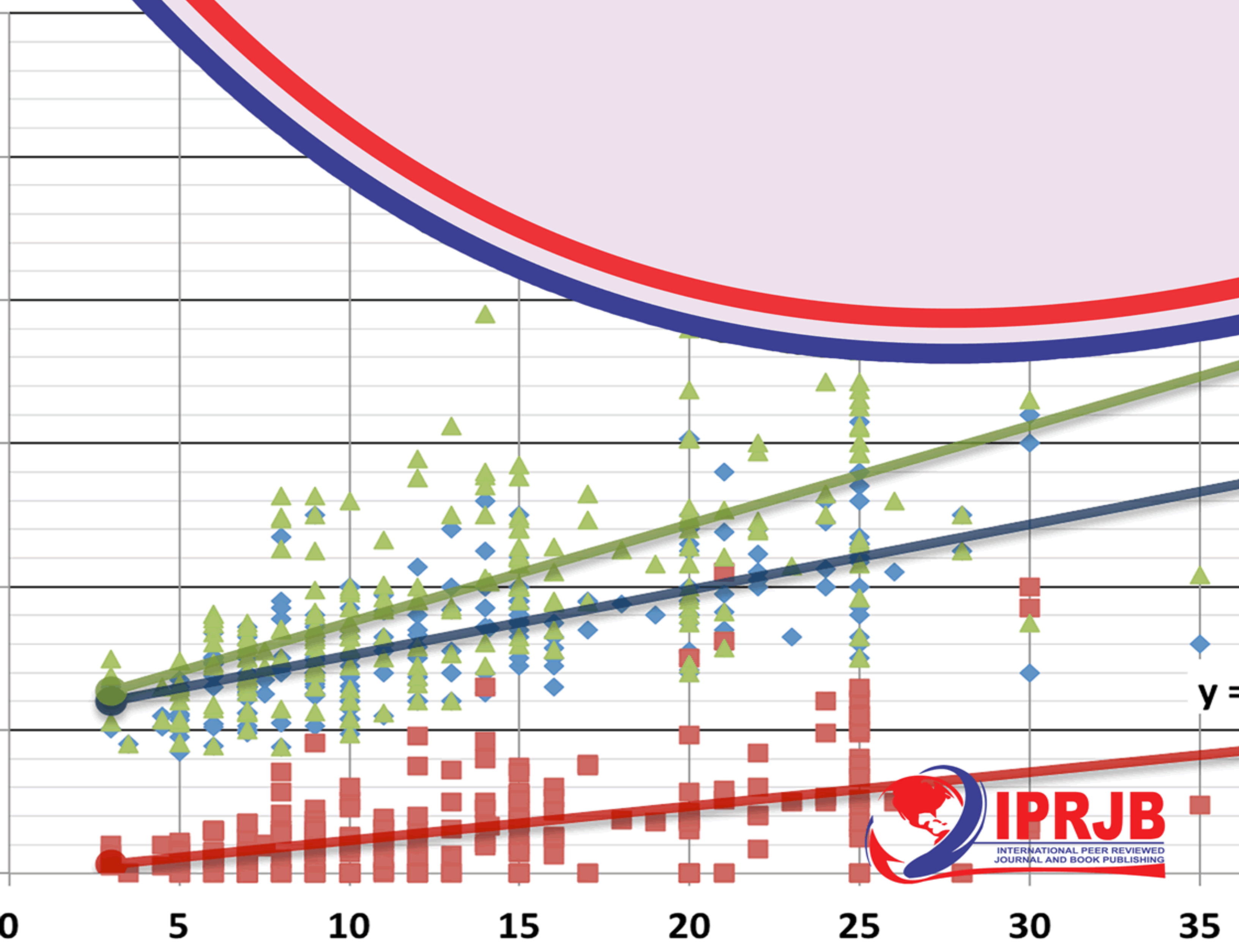

IPRJB
INTERNATIONAL PEER REVIEWED JOURNAL AND BOOK PUBLISHING

Journal of Statistics and Actuarial

Research
ISSN 2518-881X
Vol 6, Issue 1, No.1, pp 1- 19, 2022

www.iprjb.org

HOMOGENEOUS OPERATORS AND WEIGHTED SHIFT WITH MULTIPLIERS

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Abstract

In this paper we show that a homogenous operator is unitary and a reducible homogenous weighted shift is un weighted bilateral shift, also a projective representation is irreducible, and the quasi-invariant is equivalent to a unitary representation.

Journal of Statistics and Actuarial

Research
ISSN 2518-881X
Vol 6, Issue 1, No.1, pp 1- 19, 2022

INTRODUCTION

All Hilbert Spaces in this paper are separable Hilbert spaces over the field of complex numbers. The set of all unitary operators on a Hilbert space H will be denoted by $\mathcal{U}(\mathcal{H})$. When equipped with any of the usual operator topology $\mathcal{U}(\mathcal{H})$becomes a topological group. All these topologies induce the same Borel structure on $\mathcal{U}(\mathcal{H})$. We shall view $\mathcal{U}(\mathcal{H})$as a Borel group with this structure. Z, Z^{+}, Z^{-} will denote the set of all integers, non-negative integers and non-positive integers respectively, R and C will denote the Real and Complex numbers. D and T will denote the open unit disc and the unit circle in C, and $\overline{D}$ will denote the closure of D in C , Mob will denote the Mobius group of all bi holomorphic automorphisms of D. Recall that Mob = { $\varphi\alpha, \beta \ \in T, \beta \in D)$ } , where :

$$\varphi_{\alpha\beta}(Z) = \alpha \frac{z-\beta}{1-\beta z}, \ z \in D. \qquad (1.1)$$

Mob is topologies via the obvious identification with TxD. With this topology, Mob becomes a topological group. Abstractly, it is isomorphic to PSL (2, R) and to *PSU* (1.1).

Lemma (1):

If T is a homogenous operator such that T^k is unitary for some positive integer k then T is unitary.

Proof:

Let $\varphi \in$Mobs since $\varphi(T)$ is unitary, it follow that $(\phi(T))^k$ is unitary equivalent to T^k and hence is unitary $I_n$particular taking $\varphi = \varphi_\beta$we find that the inverse and the adjoin of $(T-\beta)^k (I-\overline{\beta}T)^{-1}$ are equal $(T-\beta I)^{-k}(I-\overline{\beta}T)^k$.

Since T^k is unitary implies that $(T-\beta I)^{-k}(I-\overline{\beta}T)^k = (T^*-\overline{\beta}I)^k (I-\overline{\beta}T)^k$ and we get $(T^*-\overline{\beta}I)^k (I-\beta T^*)^{-k}$ and hence $T^*T = I$ we have $(I-\overline{\beta}T)^k (I-\beta T^*)^k = (T-\beta I)^k (T^*-\overline{\beta}I)^k$.

For all $\beta \in D$ the two side of this equation is expanding binomially and the binomial rule is

$$(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r$$

By applying this rule we get

$$\left(\sum_{m=0}^{k}(-1)^m \binom{k}{m} \beta^{-m} T^m\right)\left(\sum_{n=0}^{k}(-1)^n \binom{k}{n} \beta^n T^{*n}\right) = \sum_{m=0}^{k}\sum_{n=0}^{k}(-1)^m (-1)^n \binom{k}{m}\binom{k}{n} \beta^{-m}\beta^n T^m T^{*n}$$

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Research
ISSN 2518-881X
Vol 6, Issue 1, No.1, pp 1- 19, 2022

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$$= \sum_{m,n=0}^{k} (-1)^{m+n} \binom{k}{m}\binom{k}{n} \beta^{-m} \beta^{n} T^{m} T^{n}$$

$$= \sum_{m,n=0}^{k} (-1)^{m+n} \binom{k}{m}\binom{k}{n} \beta^{-m} \beta^{n} T^{k-n} T^{*k-n}$$

by equaling the coefficients of powers weight

$T^{*n} T^{m} = T^{k-n} T^{*k-n}$ for $0 \le m, n \le k$

Noting that our hypothesis on T implies that T is invertible, we find $\frac{T^{m}}{T^{k-m}} = \frac{T^{*k-m}}{T^{*n}}$ is implies $T^{m+n-k} = T^{*k-m-n}$ for all m,n in this range, in particular taking $m+n=k-1$ we have $T^{-1} = T^{*}$ this T is unitary.

Theorem (2):

Up to unitary equivalence, the only reducible homogenous weighted shift (with non-zero weights) is the un weighted bilateral shift B

Proof:

Any such operator T is a bilateral shifts and its weight sequence W_n, $n \in z$ is periodic say with period, we may assume $W_n > 0$ for all n in z

The spectral radius $r(T)$ of T is given by the following

$$r^{+} = \lim_{n\to\infty} \left[\underset{j=0}{Sup} \left(\omega_j \omega_{j+1} \cdots \omega_{n+j-1} \right) \right]^{\frac{1}{n}}, \; r(T) \max(\bar{r}, r^{+}) \text{ where}$$

$$r^{+} = \lim_{n\to\infty} \left[\underset{j=0}{\underset{j\ge 0}{Sup}} \left(\omega_j \omega_{j+1} \cdots \omega_{n+j-1} \right) \right]^{\frac{1}{n}} \text{ And } \bar{r} = \lim_{n\to\infty} \left[\underset{j=0}{\underset{j<0}{Sup}} \left(\omega_{j-1} \omega_{j-2} \cdots \omega_{j-n} \right) \right]^{\frac{1}{n}}$$

In our case since the weight sequence ω_n is periodic with period k this formula for the spectral radius reduces to

$$r(T)=(\omega_0\omega_1\dots\omega_{k-1})^{\frac{1}{k}}$$

Now assume that T is also homogenous, then $r(T)=1$. Thus $\omega_0\omega_1\dots\omega_{k-1}$ by the periodicity of the weight sequence, it then follows that $\omega_n\omega_{n+1}\dots\omega_{n+k-1}=1\ \forall_n\in Z$ therefore it $x_n, n\in z$ is the orthogonal basis such that $Tx_n=x_{n+k}=B^k x_n$ for all n and hence $T^k=B^k$, since B is unitary show that T^k is unitary therefore T is unitary. Hence $\omega_n=\|Tx_n\|=\|T\|\,\|x_n\|$ since $\|T\|=1$ implies $\|x_n\|=1$ for all n. Thus $T=B$.

Definitions (3):

If T is an operator on a Hilbert space $\mathcal{H}$ then a projective representation π of Mobius on $\mathcal{H}$ is said to be associated with T if the spectrum of T is contained in D and

$$\phi(T)=\pi(\phi)^* T\pi(\phi) \qquad (1)$$

For all elements φ of Mob

Theorem (4):

If T is an irreducible homogenous operator ,then T has a projective representation of Mob associated with it- Further this representation is uniquely determined by T.

For any projective representation π of Mobs let $\pi^\#$ denote the projective representation of Mobs obtained by composing with the automorphism * of Mobs so

$$\pi^\#(\phi)=\pi(\phi^*) \qquad (2)$$

We note.

Proposition (5):

If the projective representation π associated with a homogenous operator T then $\pi^\#$ is associated with the adjoin T^* of T. Further T is invertible then $\pi^\#$ is associated with T^{-1} also it is follows that T and T^{*-1} have the same associated representation .

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Theorem (6):

Let $\mathcal{H}$ be a Hilbert space of function on Ω such that the operator T on $\mathcal{H}$ giver by $(Tf)(x) = xf(x), x \in \Omega,\ f \in \mathcal{H}$ is bounded. Suppose these are a multiplier representation π of Mob on $\mathcal{H}$. Then T is homogenous and π is associated with T .

Definition (7):

Let T be a bounded operator on a Hilbert space $\mathcal{H}$ then T is called a block shift is there is an orthogonal decomposition $\mathcal{H} = \oplus_{n \in \mathcal{I}} \omega_n$ of $\mathcal{H}$ in to non-trivial subspace $\omega_n,\ \ n \in I$ such that $T(\omega_n) \subseteq \omega_{n+1}$ the following is due to Mark Ordower.

Lemma (8):

If T is an irreducible block shift then the blocks of T are uniquely determined by T .

Proof:

Fix an element $\alpha \in T$ of infinite order and let $V_n, n \in I$ be blocks of T then define a unitary S_1 operator S by $Sx = \alpha^n x$ for $x \in V_n,\ n \in I$. Notice that by our assumption on α the eigen value $\alpha^n, n \in I$ of S are distinct and the blocks V_n of T are precisely the eigen spaces of S . If ω_n , $n \in J$ are also blocks of T then define of other unitary S_1 replacing the blocks V_n the blocks ω_n by the blocks the definition of S .

A simple computation shows that we have $STS^* = S_1 T S_1^*$ hence $S_1^* S$ commutes with T since $S_1^* S$ is unitary and T is irreducible and $S_1^* S$ is a scalar. That is $S_1 = \beta S$ for $\beta \in T$ therefore S has same eigen spaces as S thus the blocks of T are uniquely determined as eigen spaces of S .

To define the projective representation and multipliers, let G to be a locally compact second countable to topological group then a measurable function.

$$\pi: G \to u(\mathcal{H})$$

Is called a projective representation of G on the Hilbert space $\mathcal{H}$ if there is function $m: G \times G \to T$ such that

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Research
ISSN 2518-881X
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$$\pi(1)=1,\ \pi(\ g_1 g_2)m(g_1 g_2)\ \ \pi(g_1)\pi(g_2) \quad (3)$$

Forall $(\ g_1,g_2)G$. Two projective representation $\pi\ ,\pi\ _2$ in the Hilbert spaces $\mathcal{H}_1\ ,\mathcal{H}_2$ will be called the equivalent if there is exists a unitary operator $u:\mathcal{H}_1\to\mathcal{H}_2$, and function $\gamma:G\to T$. Such that $\pi_2(g)\alpha(g)U\pi_1(g)$. For all $(g)\in G$ we shall identify two projective representation they are equivalent.

Recall that a projective representation π of G is called irreducible if the unitary operator $\pi(g)$, $g\in G$have no common non-trivial reducing subspace. Clearly $m:G\times G\to T$ is a Borel map. In view of equation (3) m satisfies $m(g,1)=1=m(1,g)$

$$m(g_1 g_2)m(g_1,g_2,g_3)=m(g_1,g_2,g_3)m(g_2,g_3) \quad (4)$$

Proof equation (4) :

From equation (6) $\pi(g_1,g_2)=m(g_1,g_2)\pi(g_1)\pi(g_2)$ which implies that

$$m(g_1,g_2)=\pi(g_1,g_2)\pi(g_1)\pi(g_2)$$

Then

$$m(g,1)=\pi(g)/\pi(g)\pi(1)=1$$

$$m(1,g)=\pi(g)/\pi(1)\pi(g)=1$$

And

$m(g_1,g_2,g_3)=\pi(g_1,g_2,g_3)/\pi(g_1,g_2)\pi(g_3)$ the left hand side of equation. (4)

$$m(g_1,g_2)m(g_1,g_2,g_3)=\frac{\pi(g_1 g_2)}{\pi(g_1)\pi(g_2)}\cdot\frac{\pi(g_1 g_2 g_3)}{\pi(g_1 g_2)\pi(g_3)}=\frac{\pi(g_1 g_2 g_3)}{\pi(g_1)(g_2)\pi(g_3)}$$

And the right hand side

$$m(g_1,g_2 g_3)m(g_2,g_3)=\frac{\pi(g_1 g_2 g_3)}{\pi(g_1)\pi(g_2 g_3)}\cdot\frac{\pi(g_2 g_3)}{\pi(g_2)\pi(g_3)}=\frac{\pi(g_1 g_2 g_3)}{\pi(g_1)\pi(g_2)\pi(g_3)}$$

$$m(g_1,g_2)m(g_1 g_2,g_3)=\pi(g_1,g_2 g_3)\pi(g_2,g_3)$$

Journal of Statistics and Actuarial Research
ISSN 2518-881X
Vol 6, Issue 1, No.1, pp 1- 19, 2022

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For all group of elements g, g_1, g_2, g_3 any Borel function m into T satisfying (4) is called a multiplier in the group.

Definition (9):

Two multipliers m and on the group G are called equivalent I there is Borel function $\gamma : G \to T$ such that $\gamma(g_1, g_2) g\tilde{m}(g_1, g_2) = \gamma(g_1)\gamma(g_2) m(g_1, g_2)$ for all $g_1, g_2 \in G$ and clearly equivalent projective reorientation have multipliers, the multipliers equivalent to the trivial multiplier are called exact. The exact multipliers form a subgroup of the multiplier group, the quotient is called the second co homology group $H^2(G, T)$ we shall need .

Theorem (10):

Let G be a connected semi-simple lie group then every projective representation of G is a direct. Integral of irreducible projective representation of G .

Proof:

Let π be a projective representation of G let $\bar{G}$ be the universal cover of G and let $P : \bar{G} \to G$ be the covering homomorphism. Define projective representation π_0 of $\bar{G}$ by $\pi_0(\tilde{x}) = \pi(x)$ where $x = P(\tilde{x})$ a trivial computation of $\bar{G}$ and its multiplier m_0 is given by $m_0(\tilde{x}, \tilde{y}) = m(x, y)$ where $x = P(\tilde{x}),\ y = P(\tilde{y})$.

However since $\bar{G}$ is a connected Lie group $H^2(\tilde{G}, T)$ is trivial therefore m_0 is exact that is a Borel function

$\gamma : \tilde{G} \to T$

Such that

$$m(x, y) = m_0(\tilde{x}, \tilde{y}) = \gamma(\tilde{x})\gamma(\tilde{y}) / \gamma(\tilde{x}\tilde{y}) \qquad (5)$$

For all $\tilde{x}\tilde{y} \in \tilde{G}$, and $x = P(\tilde{x}),\quad y = P(\tilde{y})$

Now we define the ordinary representation $\tilde{\pi}$ of $\tilde{G}$ by $\tilde{\pi}(\tilde{x}) = \alpha(\tilde{x})\pi_0(\tilde{x})$ for $\tilde{x} \in \tilde{G}$ the ordinary representation $\tilde{\pi}_t$ of $\tilde{G} : \tilde{\pi}(\tilde{x} =) = \int^{\oplus} \tilde{P}_t i(\tilde{x}) dp(t)$, $\tilde{x} \in \tilde{G}$ replacing $\tilde{\pi}$ its definition in term of π , we get that for each $x \in G$, $\pi(x) = \int^{\oplus} \gamma(\tilde{x})^{-1} \hat{\pi}_t dp(t)$ for any $\tilde{x}$ such that $x = P(\tilde{x})$. So we would like to define $\pi_t : G \to u(\mathcal{H})$ by $\pi_t(x) = \gamma(\tilde{x})^{-1} \tilde{\pi}_t(\tilde{x})$ for any $\tilde{\pi}$ as above and verity that π_t thus defined is an irreducible projective representation of G with multiplier m. But first we must show that π_t is well defined, that is if $\tilde{x}, \tilde{y}$ are elements of mapping in the same element x of G under P the we need to show

$$\gamma(\tilde{x})^{-1} \tilde{\pi}_t(\tilde{x}) = \gamma(\tilde{y})^{-1} \tilde{\pi}_t(\tilde{y}) \tag{6}$$

Let $\tilde{Z}$ be the kernel of the covering map P. Since $\tilde{Z}$ is a discrete normal subgroup of the connected topological group $\tilde{G}, \tilde{Z}$ is a central subgroup of $\tilde{G}$. Since for each $t, \tilde{\pi}_t$ is irreducible it follows that there is a Borel function [44]. $\gamma_t : \tilde{Z} \to T$. Such that $\tilde{\pi}(\tilde{Z}) = \gamma_t(\tilde{Z}) I$ for all $\tilde{z} \in \tilde{Z}$ we have $\tilde{\pi}(\tilde{Z}) = Z(\tilde{Z}) \pi_0(\tilde{Z}) = \gamma(\tilde{Z}) \pi(1) = \gamma(\tilde{Z}) I$ for all $\tilde{z} \in \tilde{Z}$.

Therefore evaluating $\tilde{\pi}(\tilde{z})$ using its t all in a set of full P measure and all $\tilde{z} \in \tilde{Z}$. Replacing the domain of integration by this subset if need be we may assume that $\gamma_t = \gamma$ for all t. Thus

$$\tilde{\pi}(\tilde{z}) = \gamma(\tilde{z}) I \tag{7}$$

for all $\tilde{z} \in \tilde{Z}$ and for all t. Also for $\tilde{x} \in \tilde{G}$ and $\tilde{z} \in \tilde{Z}$ we have

$$\gamma(\tilde{x}) r(\tilde{Z}) / r(\tilde{x}\tilde{Z}) = m(\tilde{x}, \tilde{Z}) = m(x, 1) = 1$$

where $x = P(\tilde{x})$ and hence

$$\gamma(\tilde{x}\tilde{Z}) = \gamma(\tilde{x}) \gamma(\tilde{Z}) \tag{8}$$

Now we come back to proof equation (6)

Since $P(\tilde{x}) = P(\tilde{y})$, there is $\tilde{z} \in \tilde{Z}$ such that $\tilde{y} = \tilde{x}\tilde{Z}$ using equation (6) we get

Journal of Statistics and Actuarial

Research
ISSN 2518-881X
Vol 6, Issue 1, No.1, pp 1- 19, 2022

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$\gamma(\tilde{y})^{1}\tilde{\pi}_t(\tilde{y})=\gamma(\tilde{x})^{-1}\gamma(\tilde{Z})^{-1}\tilde{\pi}_t(\tilde{x})\tilde{\pi}_t(\tilde{Z})$ from equation (8) we have $\gamma(\tilde{y})^{1}\tilde{\pi}_t(\tilde{y})=$ $\gamma(\tilde{x})^{-1}\tilde{\pi}_t(\tilde{x})$ this proves equation (6) and hence π_t shows is well defined. Now for $x,y\in G\ \pi_t(xy)=\gamma(\tilde{x}\tilde{y})\tilde{\pi}_t(\tilde{x}\tilde{y})$

We apply $\tilde{\pi}_t(\tilde{x}\tilde{y})=\tilde{\pi}_t(\tilde{x})\tilde{\pi}_t(\tilde{y})$

We get $\pi_t(xy)=\gamma(\tilde{x}\tilde{y})\tilde{\pi}_t(x)\tilde{\pi}_t(y)$

We use $\pi_t(x)=\gamma(x)^{-1}\tilde{\pi}_t(\tilde{x})$

And $\pi_t(x)=\gamma(\tilde{y})^{-1}\tilde{\pi}_t(\tilde{x})$

This implies $\tilde{\pi}_t(\tilde{x})=\pi_t(x)/\gamma(\tilde{x})^{-1}$

$\tilde{\pi}_t(\tilde{y})=\pi_t(y)/\gamma(\tilde{y})^{-1}$

by applying eq. (8) we get

$$\pi_t(xy)=\gamma(\tilde{x}\tilde{y})\frac{\pi_t(xy)}{\gamma(\tilde{x})}.\frac{\pi_t(xy)}{\gamma(\tilde{y})}=\frac{\gamma(\tilde{y})r(\tilde{y})\pi_t(x)\pi_t(y)}{\gamma(\tilde{x})^{-1}\gamma(\tilde{y})^{-1}}=\frac{\gamma(\tilde{x})r(\tilde{y})}{\gamma(\tilde{x}\tilde{y})}\pi_t(x)\pi_t(y)$$

form eq. (8) we get

$$\frac{\gamma(\tilde{x})\gamma(\tilde{y})}{\gamma(\tilde{x}\tilde{y})}\pi_t(x)\pi_t(y)=m_0(\tilde{x},\tilde{y})\pi_t(x)\pi_t(y)$$

Since $m_0(\tilde{x},\tilde{y})=m(x,y)$ then $\pi_t(xy)=m(x,y)\pi_t(x)\pi_t(y)$ where $\tilde{x},\tilde{y}\in\tilde{G}$ are such that $x=P(\tilde{x})$, $y=P(\tilde{y})$ this shows that π_t is indeed projective

Representation of G will multiplier m. Since from the definition of π_t it is clear that π_t and $\tilde{\pi}_t$ have the same invariant subspaces and since the latter is irreducible it follows that each π_t is irreducible. Thus we have the required decomposition of π as a direct integral of irreducible projective representation π_t with the same multiplier as $\pi:\pi=\int^{\oplus}\pi_t dp(t)$. As a consequence of theorem (1-10) we have the following corollary, here as above $\tilde{G}$ in the universal cover of

G, $P:\tilde{G}\to G$ is the covering map. Fix a Borel section $S:G\to\tilde{G}$ for P such that $S(1)=1$. Notice that the kernel $\tilde{Z}$ of P is naturally identified with the fundamental a group $\pi^1(G)$ of G. Define the map .

$$\alpha: G\times G\to\tilde{Z} \text{ by } \alpha(x,y)=S(xy)S(y)^{-1}S(x)^{-1},\quad x,y\in G \qquad (9)$$

For any character (i.e., continuous homomorphism into the circle group T) of $\pi^1(G)$ define $m_x: G\times G\to T$ $m_x(x,y)=x(\alpha(x,y))$, $x,y\in G$. Since $\tilde{Z}$ is a central subgroup of $\tilde{G}$ it is easy to verity that α satisfies the multiplier identity .

Hence m_x is a multiplier on G for each character x of $\tilde{Z}$.

Corollary (11):

Let G be a connected semi-simple Lie group, then the multiplier m_x are mutually in equivalent and every multiplier on G is equivalent to m_x for a unique characteristic x. In other words $x\to[m_x]$ defines a group isomorphism $H^2(G,T)\equiv HomH,(G,T)$.

for $\varphi\in$ MÓb, φ is non-vanishing analytic on $\overline{D}$. Hence there is an analytic branch of log φ^1 on D' Fix such a branch for each φ such that

(a) For $\varphi=1$, $\log\varphi'=0$

(b) The map $(\varphi,z)\to\log\varphi'(z)$ from MÓb $\times\overline{\text{D}}$ into □ is a Borel function with such a determination of the logarithm we define the function $(\varphi')^{\frac{N}{2}}$ and $N>0$ and $\arg\varphi'$ on D' by $\varphi(\varphi')^{\frac{N}{2}}=\exp\left(\frac{N}{2}\log\varphi'(z)\right)$, and $\arg\varphi'(z)=\operatorname{Im}\log\varphi'(z)$ for $n\in z$ let $f_n:T\to T$ defined by $f_n(z)=Z^n$ in the following all the Hilbert space $\mathcal{H}$ is spanned by orthogonal of set $\{f_n:n\in I\}$. Where is some subset of Z thus the Hilbert space of functions is specified by the set I and

$\{\|f_n\|, n \in I\}$ for $\varphi \in$ MÓb and complex parameters N and μ define the operator $R_{\lambda\mu}(\varphi^{-1})$on $\mathcal{H}$ by

$$R_{\lambda\mu}(\varphi^{-1})f(Z) = \varphi^{-1}(Z)^{\frac{N}{2}}|\varphi'(z)|^{\mu}(f(\varphi)(z)) \qquad z \in T,\ f \in \mathcal{H}, \varphi \in \text{MÓb}$$

We obtain a complete result of the irreducible projective representations of Mob is follows that , Holomorphic discrete series representations D_λ^+ here $\lambda > 0,\ \mu = 0, I = Z^+$ and $\|f_n\|^2 = \frac{\Gamma(n+1)\Gamma(\lambda)}{\Gamma(n+\lambda)}$ if $n = 0$ we get $\|f_n\|^2 = 0$ for $n \geq 0$ for each f in the representation space there is an $\tilde{f}$ analytic in D such that f is the non-tangential bounding value of $\tilde{f}$, by the identification the representation space may be identified with the function Hilbert space $(\mathcal{H})^{(N)}$ of analytic functions on $\mathfrak{D}$ with reproducing kernel

$$(1-2\overline{w})^{-N},\ z, w \in D.$$

Principal series representation $C_{\lambda,\delta}$ $-1 < \lambda \leq 1, s$ purely imaginary . The equation

$$\|f_n\|^2 = \frac{\Gamma(n+1)\Gamma(\lambda)}{\Gamma(n+\lambda)} = \frac{n\Gamma(n)\Gamma(\lambda)}{n\Gamma(n)}$$ Where $\lambda \leq 1$ so $\|f_n\|^2 = 1$, here $\lambda = \lambda, \mu = \frac{1-\lambda}{2} + s$, $I = Z$, $\|f_n\| = 1$ for all n and the complementary series representation $C_{\lambda,\delta}$, $-1 < \lambda < 1$, $0 < \delta < \frac{1}{2}(1-|\lambda|)$, here $\lambda = \lambda, \mu = \frac{1}{2}\left(1 - \frac{\lambda}{2}\right) + \delta$, $I = Z$ and

$$\|f_n\|^2 \prod_{k=0}^{|n|-1} \frac{k \pm \frac{\lambda}{2} + \frac{1}{2} - \delta}{k \pm \frac{\lambda}{2} + \frac{1}{2} + \delta} \quad , n \in Z$$

Where one takes the upper or lower sign according as n is positive or negative.

Theorem (12):

(i) m_ω Is a multiplier of Mobs for each $\omega \in T$ up to equivalent m_ω , $\omega \in T$ are all the multipliers in other words, H^2 (Mob) is naturally isomorphic to T via the map $\omega \mapsto m_\omega$.

(ii) For each of the representations of Mob result above.

The associated multiplier is m_{ω} where $\omega e = e^{i\pi N}$ in each case except for the auti-holomorphic discrete series, from the definition of $R_{\lambda,\mu}$ one calculates that the associated multiplier m is given by

$$m\left(\phi_1^{-1},\phi_2^{-1}\right)=\frac{\left(\left(\phi_2\phi_1\right)'(z)\right)^{\frac{\lambda}{2}}}{\left(\phi_1'(z)^{\frac{\lambda}{2}}\right)\left(\phi_1'\left(\kappa_1(z)\right)\right)^{\frac{\lambda}{2}}},z\in T$$

For any two elements φ_1,φ_2 of Mob to show this we have

$\pi(1)=1$ From equation (3) $\pi(g_1,g_2)=m(g_1,g_2)\,\pi(g_1)\pi(g_2)$ by applying equation (3) if $R_{\lambda,\mu}=\pi$ then $\left(\pi\left(\varphi_1^{-1},\varphi_2^{-1}\right)f\right)z=m\left(\varphi_1^{-1},\varphi_2^{-1}\right)\pi\left(\varphi_1^{-1}\right),\left(\varphi_2^{-1}\right)$ implies that

$$m\left(\varphi_1^{-1},\varphi_2^{-1}\right)=\frac{\left(\pi\left(\varphi_1^{-1},\varphi_2^{-1}\right)f\right)z}{\pi\left(\varphi_1^{-1}\right),\left(\varphi_2^{-1}\right)}$$

Substituted

$$R_{\lambda,\mu}=\pi,\ m\left(\varphi_1^{-1},\varphi_2^{-1}\right)=\frac{\left(R_{\lambda,\mu}\left(\varphi_1^{-1},\varphi_2^{-1}\right)f\right)z}{R_{\lambda,\mu}\left(\varphi_1^{-1}\right),\left(\varphi_2^{-1}\right)}$$

But since

$$\left(R_{\lambda,\mu}\left(\phi^{-1}\right)f\right)z=\phi'(z)^{\frac{\lambda}{2}}\left|\phi'(z)\right|^{\lambda}\left(f\phi(z)\right)$$

Implies

$$m\left(\phi_1^{-1}\phi_2^{-1}\right)=\frac{\phi_1^{-1}(z)^{\frac{\lambda}{2}}\,\phi_2^{-1}(z)^{\frac{\lambda}{2}}\left|\left(\phi_1\phi_2\right)(z)\right|^{\mu}f\left(\phi_2\left(\phi_2\right)(z)\right)}{R_{\lambda,\mu}\phi_1^{-1}R_{\lambda,\mu}\phi_2^{-1}}$$

$$=\frac{\phi_1^{1}(z)^{\frac{\lambda}{2}}\,\phi_2^{1}(z)^{\frac{\lambda}{2}}\left|\phi_1\phi\left({}_2z\right)\right|^{\mu}f\left(\phi_2\left(\phi_2\right)(z)\right)}{R_{\lambda,\mu}\left(\left(\phi_1^{1}\phi_2^{1}\right)f\right)(z)}$$

Then

Journal of Statistics and Actuarial

Research
ISSN 2518-881X
Vol 6, Issue 1, No.1, pp 1- 19, 2022

$$m\left(\phi_1^{-1}\phi_2^{-1}\right)=\frac{\phi_1(z)^{\frac{\lambda}{2}}-\phi_2(z)^{\frac{\lambda}{2}}\left|\phi_1\phi_2(z)\right|^{\mu}f\left(\phi_2(\phi_1 z)\right)}{\phi_1'(z)^{\frac{\lambda}{2}}\left(\phi_2'(\phi_1)(z)\right)^{\frac{\lambda}{2}}\left|\phi_1\phi_2(z)\right|^{\mu}f\left(\phi_2(\phi_1 z)\right)}=\frac{\left(\phi_1\phi_2\right)'(z)^{\frac{\lambda}{2}}}{\phi_1^1(z)^{\frac{\lambda}{2}}\left(\phi_2^1(\phi_1)(z)\right)^{\frac{\lambda}{2}}}$$

Notice that the right hand side of this equation is an analytic function of z in $\mathfrak{D}$ and it is of constant modulus1 in view of the chain rule for differentiation therefore by the maximum modulus principle, this formula is independent of z for $z\ in\overline{\mathfrak{D}}$. Hence we may take z = 0 in this formula and thus $m=m_\omega$ with $\omega=e^{i\pi N}$ so m is the multiplier associated with $\pi^{\#}$ is $\overline{m}$ since $\overline{D}_\lambda=D_N^{+\#}$ it follows that if $\pi=\overline{D}_\lambda$ is the anti-holomorphic discrete series, then multiplier is m_ω where $\omega e=e^{i\pi N}$. The multiplier $m_\omega, w\in T$ are naturally bioequivalent (since $w\to\left[m_\omega\right]$) is clearly a group homomorphism from T onto $H^2(\text{MÓb,T})$ this amounts to verifying that m_ω is never exact for $w\neq 1$ this fact may be deduced from corollary (1-11) as follows. Identify Mob with $T\times D\ via\ \varphi_{\alpha,\beta}\mapsto(\alpha,\beta)$ the group low on $T\times D$ is given by $(\alpha_1,\beta_1)(\alpha_2,\beta_2)=\left(\alpha_1\alpha_2.\frac{1+\bar{\alpha}_2\beta_1\bar{\beta}_2}{1+\alpha_2\bar{\beta}_1\beta_2},\frac{\beta_1+\alpha_2\beta_2}{\alpha_2+\beta_1\beta_2}\right)$, the identity in $T\times D$ is (1,0) and inverse map is $(\alpha,\beta)^{-1}=(\overline{\alpha}-\alpha\beta)$ then the universal cover is naturally identified with $R\times D$ taking covering map. $R\times D\to T\times D$ to be $P(t,\beta)=\left(e^{2\pi it},\beta\right)$, the group low on $R\times D$ is determined by the requirement that P be a group homomorphism as follows

$$(t_1,\beta_1)(t_2,\beta_2)=t+_1 t_2+\frac{1}{\pi}Im\log\left(1+e^{-2\pi it}\beta_1\bar{\beta}_2h\right)\frac{\beta_1+e^{2\pi it_2}\beta_2}{e^{2\pi it_2}+\beta_1\bar{\beta}_2}$$

To shows this we have

Let $\alpha_1=e^{2\pi it_1}$, $\alpha_2=e^{2\pi it_2}$. Substitute α_1 and α_2 in the following equation

$$(\alpha_1,\beta_1)(\alpha_2,\beta_2)=\left(\alpha_1\alpha_2.\frac{1+\alpha_2'\beta_1\overline{\beta}_2}{1+\alpha_2\beta'_1\beta_2},\frac{\beta_1+\alpha_2\beta_2}{\alpha_2+\beta_1\beta_2'}\right)$$

We get

$$(\alpha_1,\beta_1)(\alpha_2\beta_2)=\left(e^{2\pi i t_1}.e^{2\pi i t_2}.\frac{1+e^{-2\pi i t_2}\beta_1\overline{\beta}_2}{1+e^{2\pi i t_2}\overline{\beta}_1\beta_2},\frac{\beta_1+e^{2\pi i t_2}\beta_2}{e^{2\pi i t_2}+\beta_1\overline{\beta}_2}\right)$$

$$=\left(e^{2\pi i\left(t_1+t_2\right)}.\left(1+e^{-2\pi i t_2}\beta_1\overline{\beta}_2\right)\left(1+e^{2\pi i t_2}\beta_1\overline{\beta}_2\right)^{-1},\frac{\beta_1+e^{2\pi i t_2}\beta_2}{e^{2\pi i t_2}+\beta_1\overline{\beta}_2}\right)$$

$$=\left(e^{2\pi i\left(t_1+t_2\right)}.\left(1+e^{-2\pi i t_2}\beta_1\overline{\beta}_2\right)\left(1+e^{-2\pi i t_2}\beta_1\overline{\beta}_2\right),\frac{\beta_1+e^{2\pi i t_2}\beta_2}{e^{2\pi i t_2}+\beta_1\overline{\beta}_2}\right)$$

$$=\left(e^{2\pi i\left(t_1+t_2\right)}.\left(1+e^{-2\pi i t_2}\beta_1\beta_2'\right)^2,\frac{\beta_1+e^{2\pi i t_2}\beta_2}{e^{2\pi i t_2}+\beta_1\overline{\beta}_2}\right),$$

and this gives

$$\left(t_1,\beta_2\right)\left(t_2,\beta_2\right)=T_1+T_2+\frac{1}{\pi}\operatorname{Im}\log\left(1-e^{-2\pi i t_2}\beta_1\beta_2,\frac{\beta_1+e^{2\pi i t_2}\beta_2}{e^{2\pi i t_2}+\beta_1\overline{\beta}_2}\right)$$

Where (log) denote the principle branch of the logarithm on right halt plane.

The identity in $R\times D$ is (0,0) and the inverse map is $(t,\beta)^{-1}=\left(-t-e^{2\pi i t}\right)$ and the kernel $\tilde{Z}$ of the covering map P is identified with additive group Z via $n\to(n,0)$ so we choose a Borel branch of the argument function satisfying $\arg(\overline{Z})=\arg(Z)$, $z\in T$ we make an explicit choice of the Borel function $(\varphi,z)\to\arg(\varphi'(z))$ as follows $\arg\varphi'_{\alpha,\beta}(z)=\arg(\alpha)-2\operatorname{Im}\log(1-\beta z)$ let's also choose function $s:T\times D\to R\times D$ as follows $S(\alpha,\beta)=\left(\frac{1}{2\pi}(\alpha),\beta\right)$ and easy computation shows that for these choices we have $S\left(\phi_1\phi_2\right)S\left(\phi_2^{-1}\right)S\left(\phi_1^{-1}\right)=-n\left(\phi_1\phi_2\right)$ for φ_1,φ_2 in Mob. Hence we get that for $w\in T, m_w=m_\chi$ where $\chi=\chi_w$ is the character n maps to w^{-n} of Z. Thus the map $w\to[m_w]$ is but a special case of the isomorphism $\chi\to m_\chi$ of corollary (1-11) to show the simple representation of the Moby's group let k be the maximal compact subgroup of Mob given by $\{\varphi_{\alpha,0} : \alpha \in T\}$ of course k is isomorphic to the circle group T via *via* $\alpha \to \varphi_{\alpha,0}$.

Definition (13):

Let π be a projective representation of Mob and we shall say π is normalized if π / k is an ordinary representation of k.

Lemma (14):

Any projective representation δ of Mob then δ / k is projective representation of k say with multiplier m. But $H^2(k)$ so there exists a Borel function $f : k \to T$ such that $m(x, y) = \frac{f(x)f(y)}{f(xy)}$, $x, y \in k$. Extend f to a Borel function $g : \text{MÓb} \to \text{T}$. Define π by $\pi(x) = g(x)\delta(x)$, $x \in \text{MÓb}$ then π is normalized and equivalent to δ for $n \in Z$, let χ_n be the character of T given by $\chi_n(x) = x^{-n}, x \in T$ for any normalized projective representation π of Mob and $n \in Z$ let $V_n\pi = \{v \in \mathcal{H} : \pi(x)v = \chi_n(x)v_1, \forall x \in T\}$ then $\mathcal{H} = \oplus_{n \in z} V_n\pi$. The subspace $V_n(\pi)$ are usually called the k-isotopic subspaces of $\mathcal{H}$ put $d_n(\pi) = \dim V_n\pi$ and $T(\pi) = \{n \in Z : d_n(\pi) \neq 0\}$.

Theorem (15):

If T is an irreducible homogenous operator the T is a block shift. If π is a normalized representation associated with T then the blocks of T are precisely the k –isotopic subspaces.

$$V_n(\pi), \quad n \in T(\pi).$$

Proof:

If T is an irreducible block shift then the blocks of T are uniquely determined by T. Then

$$T(V_n(\pi)) \subseteq V_{n+1}(\pi) \text{For } n \in T(\pi) \tag{10}$$

Indeed since T is irreducible then equation (10) how that π is connected and $b \notin T(\pi)$ then (10) would imply that $\oplus_{n<b} V_n(\pi)$ is a non-trivial. Since is also unbounded by theorem (3-1-21) it

follows that be re-indexing, the index can be taken to be either all integer or the non-positive integers, therefore T is a block shift. So it only remains to prove (10). To do this, fix $n \in T(\pi)$ and $v \in_n (\pi)$ for $x \in k$ we have $\pi(x)v = \chi_n(x)v$. Consequently

$$\pi(x)Tv = \pi(x^{-1})^* Tv$$

$$= \pi(x^{-1})^* T(x^{-1})(\pi(x)v)$$

$$= (x^{-1}T)^* T (x^{-N}v) = x^{-((n+1))}Tv$$

So $Tv \in V_{n+1}(\pi)$., this proves (10).

Lemma (16):

Let T is any homogenous weighted shift, let be the projective representation of associated with T. Then up to equivalent π is one of the representations further:

(a) If T is a forward shift then the associated representation is holomorphic discrete series.

(b) If T is a back word shift then the associated representation is auti-holomorphic discrete series.

(c) If T is a bilateral shift then the associated representation is either principle series or complementary series.

Theorem (17):

Up to unitary equivalence the only homogenous weighted shifts are the ones.

Proof:

Let T be homogenous weighted shift. If T is reducible we are done by theorem (1-2). So assume T is irreducible then by theorem (1-4) there is a projective, representation π of Mob associated with T. By lemma (1-3) π is one of the representation. Further replacing T by T^* if necessary, we may assume that T is either a foreword or bi-lateral shift.

According π is either a homomorphic discrete series representation or a principal complementary series representation. Hence $\pi = R_{\lambda,\mu}$ for some parameters $\lambda\mu$ recall that the representation space H_π is the closed span of the function f_n, $n \in I$ where $f_n(z) = Z^n, n \in I$

and $I \in Z^+$ in the former case and $I = Z$ in the letter case the element's f_n , $n \in I$ form a complete orthogonal set of vectors in $\mathcal{H}_\pi$, but these vectors are not unit vectors . Their norms are as given before .Since T is a weighted shift with respect to the orthogonal basis of obtained $\mathcal{H}_\pi$ by normalizing $f_n s$ where are scalar $an > 0$, $n \in I$ such that

$$Tf_n = anf_{n+1} , \qquad n \in I$$

Notice that since the $f_n s$ are not normalized the numbers an are not the weights of the weighted shift T. These weights are given by follows there the adjoin T^* acts by $w_n = a\|f_{n+1}\|/\|f\|$, $n \in I$

Its follows that the ad joint act by $T^* f_n = \frac{\|f_n\|^2}{\|f_{n-1}\|^2} an - 1 f_{n-1}$, $n \in I$ where one puts $a_{-1} = 0$ in case $I = Z^+$ let M be multiplication operator on $\mathcal{H}_\pi$ define by $Mf_n = f_{n+1}$, $n \in I$.

Notice that for each representation is corresponding operator M . Also in case M is invertible M^{*-1} is also exist. Let B be a fixed but arbitrary element of D and let $\varphi_\beta = \varphi_{-1,\beta} \in$ Mob. Notice that φ_β is an involution and this simplifies the following computation of $\pi(\varphi_\beta)$ a little bit indeed a straight foreword calculation shows that for $\pi = R_{\lambda,\mu}$ we have

$$\langle \pi(\varphi_B) f_m, f_n \rangle = C(-1)^n \overline{B}^{n-m} \|f_n\|^2 \sum_{k \geq (m-n)^+} C_k(m,n) r^k , 0 \leq r \leq 1 \qquad (11)$$

where we have put $r = |\beta|^2$, $C = \varphi_\beta^1(0)^{\frac{N}{2+m}}$ and $C_k(m,n) = \begin{pmatrix} -N-\mu-m \\ k+n-\mu \end{pmatrix}\begin{pmatrix} -\mu+m \\ k \end{pmatrix}$ since π is associated with T from the following equation (4) we have $T\pi(\varphi_\beta)(I - \overline{\beta}T) = \pi(\varphi_\beta)(\beta I - T)$ we analysis the two sides of the above equation we get

$$T(\pi)(\varphi_\beta) - T\pi(\varphi_\beta)\overline{\beta}T = \pi(\varphi_\beta)\beta - \pi(\varphi_\beta)T$$

Implies

$$T\pi(\varphi_\beta) + \pi(\varphi_\beta)T = \pi(\varphi_\beta)\beta + \overline{\beta}T\pi(\varphi_\beta)T \text{ and } \overline{\beta}T\pi(\varphi_\beta)T + \pi(\varphi_\beta)T = T\pi(\varphi_\beta)T + \pi(\varphi_\beta)T$$

where m, n fix in I, we evaluate each side of the above equation at and take the inner product of the resulting vectors with we have for the instance

$$\langle T\pi(\varphi_\beta)Tf_m, f_n\rangle = \langle \pi(\varphi_\beta)Tf_m, T^* f_n\rangle = a_m \bar{a}_{n-1} \frac{\|f_n\|^2}{\|f_{n-1}\|^2} \langle \pi(\varphi_\beta) f_{m+1}, f_{n-1}\rangle$$

and similarly for the other three terms . Now substituting from equation (11)

we get $\pi(\varphi_\beta) f_{m+1}, f_{n+1} = C(-1)^n \bar{B}^{n-m} \|f_n\|^2 \sum_{k\geq(m-n+2)} C_k(m+1, n-1)r^k$, by applying equation (11) in the main equation we have

$$\langle \pi(\varphi_\beta)Tf_m, T^* f_n\rangle = a_m \bar{a}_{n-1} \frac{\|f_n\|^2}{\|f_{n-1}\|^2} C(-1)^n \bar{B}^{n-m} \|h_{n-1}\|^2 \sum_{k\geq(m-n+2)} C_k(m+1, n-1)r^k$$

by comparing with the equation (11) we get

$$a_m \bar{a}_{n-1} C(-1) \bar{B}^{n-m^n} \|f_n\|^2 \sum_{k\geq(m-n+2)} C_k(m+1, n-1)r^k =$$

$$C\,(-1)^n\,\bar{B}^{n-m}\,\|f_n\|^2 \sum_{k\geq(m-n+2)} C_k\,(m, n)\,r^k$$

where $0 \leq r \leq 1$,

$$a_m \bar{a}_{n-1} \sum_{k\geq(m-n+2)} C_k(m+1, n-1)\,r^k = \sum_{k\geq(m-n+2)} C_k(m, n)\,r^k$$

We canceling the common factor $C(-1)^{n-1} \|f_n\|^2 \bar{B}^{n-m}$ we have the following identity in the indeterminate r which obtained from the above

$$\bar{a}_{n-1} \sum_{k\geq(m-n+2)} C_k(m, n-1)\,r^k = a_m \sum_{k\geq(m-n+2)} C_k(m+1, n)\,r^k \qquad (12)$$

Taking $m = n$ in equation (12) and equating the coefficients of r' we obtain

$(n+1-\mu)a_n = (n-\mu)\bar{a}_{n-1} + 1 \; n \in I$ (13)

ISSN 2518-881X
Vol 6, Issue 1, No.1, pp 1- 19, 2022

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