Journal of
Statistics and Acturial Research
(JSAR)

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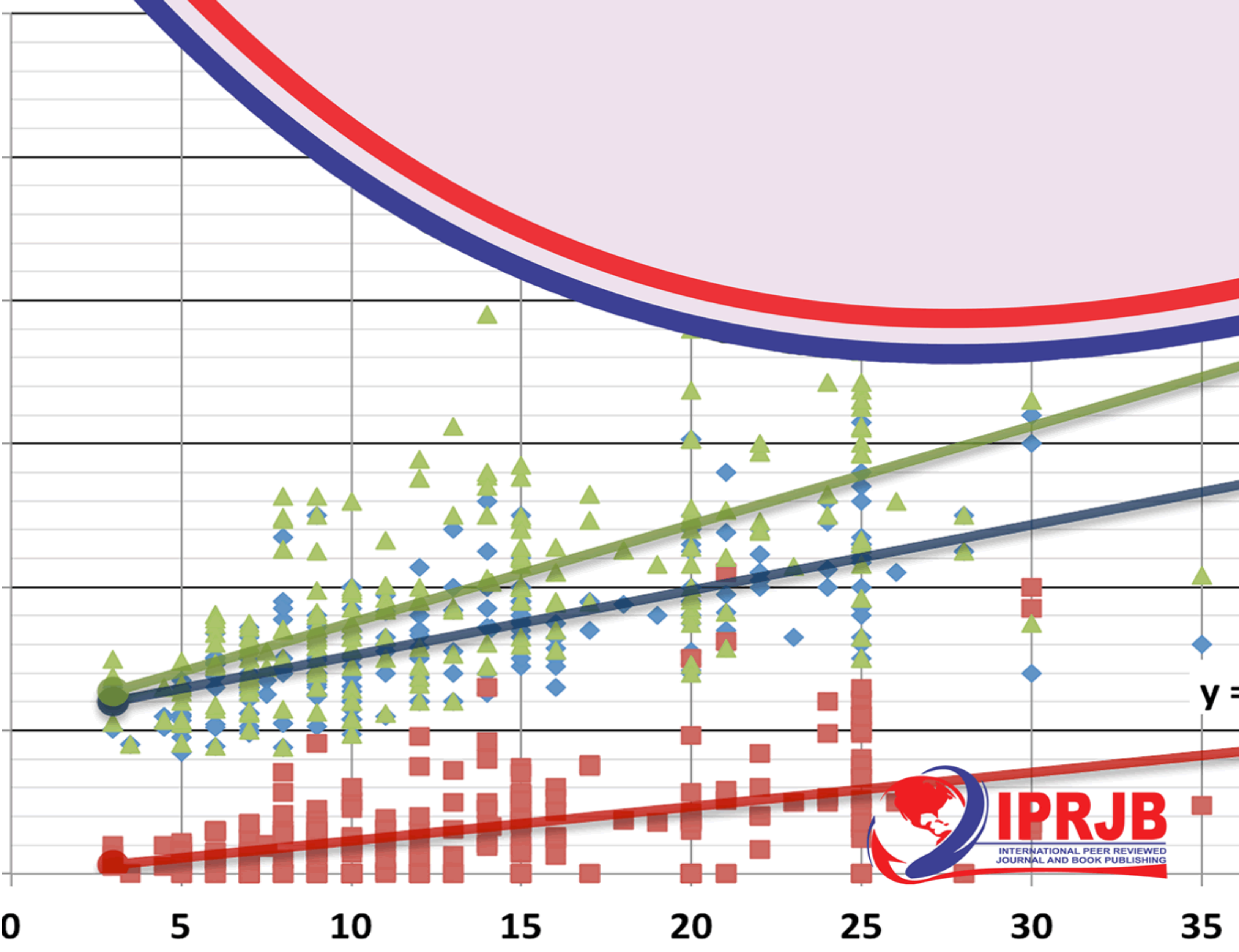

IPRJB
INTERNATIONAL PEER REVIEWED
JOURNAL AND BOOK PUBLISHING

Journal of Statistics and Actuarial

Research
ISSN 2518-881X
Vol 6, Issue 1, No.2, pp 20 - 42, 2022

www.iprjb.org

INVERSE SPECTRAL PROPARTIES FOR SYMMETRIC OPERATORS WITH WEYL FUNCTIONS

Dr. Bashir Eissa Mohammad Abedrahaman
Nyala University College of Science and Education , Sudan

Corresponding author E-mail bashireissa@Yahoo.com

Abstract

We prove that an operator measure in general is non-orthogonal and unbounded and two orthogonal spectral measures are unitarily equivalent. In accordance with the stieltjes inversion formula the spectral measure admits an analytic continuation .We discuss and prove a sharp estimate that a strictly monotone function on each component interval of the inverse function is analytic and also strictly monotone with Weyl functions.

Keywords: *symmetric operator, adjoin extensions, Nevanlinna functions, Weyl functions.*

Journal of Statistics and Actuarial

Research
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INTRODUCTION

Let S be a densely defined symmetric operator in Hilbert space $\mathcal{H}$ with deficiency indices $n_+(S) = n_-(S) \leq \infty$. We recall that abounded open interval $J = (\alpha, \beta)$ is called a gap for S if

$$\|2S - (\alpha, \beta)\| \geq (\alpha - \beta)\|f\|, f \in dom\ s , \quad (1)$$

if $\alpha \to -\infty$, then (1) turns into $(Sf, f) \geq \beta\|f\|^2$ for all $f \in dom\ S$, meaning that $(-\infty, \beta)$, is a gap for A if S is semi bounded below with the lower bound β.

Theorem (1):

Let $\{S_k\}_{k=1}^{\infty}$ be a family of closed symmetric operators S_k, defined in the separable Hilbert space R such that the operators S_k are unitarily equivalent to a closed symmetric operator A in h with equal positive deficiency indices. If there exists a boundary triple $\Pi_0 = \{\mathcal{H}_0, \Gamma_0^0, \Gamma_1^0\}$ for A^* such that the corresponding Weyl function $M(.)$ is monotone with respect to open set $J \subseteq \rho(A_0)$, $A_0 = A^* | \ker(\Gamma_0^0)$, then for any auxiliary self-adjoint operator R in some separable Hilbert space R the closed symmetric operator S admits a self-adjoin extension $\tilde{S}$ such that the spectral, parts $\tilde{S}_J$ and R_J are unitarily equivalent i.e. $\tilde{S}_J \cong R_j$ [95.109,110].

The following result is known as a generalized Nuimark dilation theorem.

Proposition (2):

If $\sum(.) : B(R) \to [\mathcal{H}]$ is a bounded operator measure, then there exist a Hilbert space k abounded operator $k \in [\mathcal{H}, K]$ and an orthogonal measure

$E(.) = B(R) \to [k]$ (an orthogonal dilation) such that

$$\sum(\delta) = k^* E(\delta) k, \delta \in B(R) \quad (2)$$

If the orthogonal dilation is minima i.e.,

$$span\{E(\delta) ran(k) : \delta \in B(R)\} = K , \quad (3)$$

then it is uniquely determined up to unitary equivalence that is if one has two bounded operator $k \in [\mathscr{H}, k]$ and $K' \in [\mathscr{H}, K]$ as well as two minimal orthogonal dilation $E(.) = B(R) \to [K]$ and $E'(.): B(R) \to [K']$ obeying $\sum(\delta) = K^* E(\delta) K \ = K'^* E'(\delta) K', \delta \in (R) B(R)$, then there exists an isometry $v: K' \to K$ such that $E'(\delta) = v^* E(\delta) v, \delta \in B(R)$.

Definition (3):

We call $E(.)$ satisfying (2) and (3) the minimal orthogonal measure associated to $\sum(.)$, or the minimal orthogonal dilation of $\sum(.)$.

Every operator measure $\sum(.)$ admits the Lebesque Jordan decomposition $\sum = \sum^{ac} + \sum^{s}, \sum^{s} \ = \sum^{sc} \ + \sum^{pp}$ where $\sum^{ac}$, $\sum^{s}$, $\sum^{sc}$ and $\sum^{pp}$ are the absolutely continuous, singular, singular continuous and pure point components (measure) of $\sum(.)$, respectively. Non-topological supports of mutually disjoint, therefore if an operator measure $\sum$ is orthogonal, $\sum(.) = E_T(.)$, then the ortho-projections $p^\tau = E_T^\tau(R)(\tau \in \{ac, sc, pp\})$ are pair wise orthogonal. Every subspace $h_T^\tau = p^\tau h$ reduces the operator $T \ = T^*$ and the Lebesgue-Jordan decomposition yields

$$
\begin{aligned}
h &= h_T^{ac} \oplus h_T^{sc} \oplus h_T^{pp} \\
T &= T^{ac} \oplus T^{sc} \oplus T^{pp}
\end{aligned}
\tag{4}
$$

Where $T^\tau = P^\tau T \uparrow h_T^\tau, \quad T \in \{ac, sc, pp\}$. Now we show Nevanlinna functions:

Let $\mathscr{H}$ be a separable Hilbert space, we recall that an operator-valued function $F: c_+ \to [\mathscr{H}]$ is said to be a Neranlinna (or Herglotz or $R_{\mathscr{K}}$) one if it is holomerphic and takes values in the set of dissipative operators on $\mathscr{H}$ i.e.,

$$\bar{S}m(F(z)) = \frac{F(z) - F(z)^*}{2!} \geq 0, \, z \in C_+$$

Usually one considers a continuation of F in □ by setting $F(z) = F(\overline{z}), z \in C_-$. Bounded operator $k \in [\mathscr{H}, K]$ obeying $\ker(K) = \ker l \sum_F{}^0(R)$ and $\sum_F^0(\delta) = k^* E_F(\delta) k, \delta \in B(R)$. By

$$\sum_F(\delta) = \int_\delta (1+t^2) d \sum_F^0(t), \delta \in B_b(R) \tag{5}$$

One defines and operator measure which in general is non-orthogonal and unbounded. It is called the unbounded spectral measure of $F(.)$. Using $\sum_F$ the representation [118],

$$F(z) = C_0 + C_1 z + \int_{-\infty}^{+\infty} \left(\frac{1}{t-z} - \frac{1}{1+t^2} \right) d \sum_F(t), z \in C_+ \cup C_- \tag{6}$$

To show this we have

From this representation $F(z) = C_0 + C_1 z + \int_{-\infty}^{+\infty} \frac{1+tz}{t-z} d \sum_F^0(t), z \in C_+ \cup C_-$. To prove representation (6) use equation (5)

$$\sum_F(\delta) = \int_\delta (1+t^2) d \sum_F^0(t), \delta \in B_b(R)$$

so $d \sum_F(\delta) = (1+t^2) d \sum_F^0(t)$, which implies that $d \sum_F^0(t) = \frac{1}{1+t^2} d \sum_F(t)$, put this in the representation above we have

$$F(z) = C_0 + C_1 z + \int_{-\infty}^{+\infty} \frac{1+tz}{t-z} \left(\frac{1}{1+t^2} \right) d \sum_F(t) = C_0 + C_1 z + \int_{-\infty}^{+\infty} \frac{1+tz}{(t-z)(1+t^2)} \left(\frac{1}{1+t^2} \right) d \sum_F(t)$$

To analysis this component we use this $\frac{1+tz}{(t-z)(1+t^2)} = \frac{A}{t-z} + \frac{Bt}{1+t^2} + \frac{c}{1+t^2} = 1+tz$ and $A(1+t^2) + Bt(t-z) + C(t-z) = 1+tz$ put $t = z$ we get $A(1+z^2) = 1+z^2$, so

$A=1$ at $t=0, A-Cz=1$ implies that $c=0$ since $A=1, c=0$. Our equation become $1+t^2+B_t(t-z)+0=1+tz$, $Bt(t-z)=1+tz-1-t^2=-t(t-z), Bt=-t\left(\frac{t-z}{t-z}\right)$, $B=-1$.

Substituted A, B, and C the equation

$$\frac{1+tz}{(t-z)(1+t^2)}=\frac{A}{t-z}+\frac{Bt}{1+t^2}+\frac{C}{1+t^2}=1+tz$$

We get the following

$$F(z)=C_0+C_1+\int_{-\infty}^{+\infty}\left(\frac{1}{t-z}-\frac{t}{1+t^2}\right)d\sum_F(t) \tag{7}$$

$z\in C_+\cup C_-$

Which complete the proof. From representation

$$F(z)=C_0+C_1z=\int_{-\infty}^{\infty}\frac{1-tz}{t-z}d\sum_F{}^0(t), z\in C_+\cup C_-$$

F determines uniquely the unbounded spectral measure $\sum_F(.)$ by means of the Stieltjes inversion formula, which is given by

$$\sum_F((a,b))=s-\lim_{\delta\to+0}s-\lim_{\varepsilon\to+0}\frac{1}{\pi}\int_{a+\delta}^{b-\delta}Sm(F(x+i\varepsilon))dx \tag{8}$$

By supp (F) we denote the topological (minimal closed) support of the spectral measure $\sum_F$. Since supp (F) is closed the set $O_F=R\setminus\text{supp}(F)$ is open. The Nevanlinna function $F(.)$ admits an analytic continuation to O_F given by

$$F(\lambda)=C_0+C_1\lambda+\int_{-\infty}^{+\infty}\left(\frac{1}{t-\lambda}-\frac{t}{1+t^2}\right)d\sum_F(t), \lambda\in O_F$$

Using this representation we immediately find that $F(.)$ is monotone on each component interval $\Delta\, of\ O_F$ i.e., $F(\lambda) \le F(\mu), \lambda < \mu, \quad \lambda, \mu \in \Delta$. In general, this relation is not satisfied if λ and μ belong to different component interval.

Definition (4):

Let $F(.)$ be a Nevanlinna function, the Nevanlinna function is monotone with respect to the open set $J \le O_F$ if for any two component intervals J_1 and J_2 of J one has $F(\lambda_1) \le F(\lambda_2)$ for all $\lambda_1 \in J_1$ and $\lambda_2 \in J_2$ or $F(\lambda_1) \ge F(\lambda_2)$ for all $\lambda_1 \in J_1$ and $\lambda_2 \in J_2$.

Let $L \in N \bigcup \infty$ be the number of component interval of J. obviously if $F(.)$ is monotone with respect to J and $L < \infty$, then there exists an enumeration $\{J_k\}_{k=1}^{L}$ of the components of J such that

$$F(\lambda_1) \le F(\lambda_2) \le \dots \le F(\lambda_L)$$

Holds for $\{\lambda_1, \lambda_2, \dots, \lambda_L\} \in J_1 \times J_2 \times \dots \times J_L$. If $L = \infty$, then it can happen that such an enumeration does not exist. If $F(.)$ is a scalar Nevanlinna function, then $F(.)$ is monotone with respect to J if and only J if the condition $F(J_1) \bigcap F(J_2) = 0$ is satisfied for any two component intervals J_1 and J_2 of J .

Definition (5):

A triple $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ consisting of an auxiliary Hilbert space $\mathcal{H}$ and linear mappings $\Gamma_i : dom(A^*) \to \mathcal{H}, i = 0,1$. Called a boundary triple for the adjoint operator A^* of A if the following two conditions are satisfied:

(i) The second Green's formula takes place

$$(A^* f, g) - (f, A^* g) = (\Gamma_1 f, \Gamma_0 g) - (\Gamma_0 f, \Gamma_1 g), f, g \in dom(A^*)$$

Journal of Statistics and Actuarial

Research
ISSN 2518-881X
Vol 6, Issue 1, No.2, pp 20 - 42, 2022

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(ii) The mapping $\Gamma=\{\Gamma_0,\Gamma_1\}: dom(A^*)\to \mathcal{H}\oplus\mathcal{H}$, $\Gamma f=\{\Gamma_0 f,\Gamma_1 f\}$ is subjective the above definition allows one to describe the set Ext_A in the following way.

Proposition (6):

Let $\Pi=\{\mathcal{H},\Gamma_0,\Gamma_1\}$ be a boundary triple for A^* then the mapping Γ established objective correspondence $\tilde{A}\to\theta=\Gamma\left(dom\left(\tilde{A}\right)\right)$ between the set Ext_A of self-adjoin linear relations in $\mathcal{H}$.

By proposition (6) the following definition is natural .

Definition (7):

Let $\Pi=\{\mathcal{H},\Gamma_0,\Gamma_1\}$ be a boundary triple for A^*. We put $A_\theta=\tilde{A}$, if $\theta=\Gamma\left(dom\left(\tilde{A}\right)\right)$ that is

$A_\theta=A^*|D_\theta$,

$$dom(A_\theta)=D_\theta=\left\{f\in dom(A^*):\{\Gamma_0 f,\Gamma_1 f\}\in\theta\right\} \qquad (9)$$

If $\theta=G(B)$ is the graph of an operator $B=B^*\in C(\mathcal{H})$, then $dom(A_\theta)$ is determined by the equation $dom(A_B)=D_B=\ker(\Gamma_1-B\Gamma_0)$. We set $A_B=A_\theta$

Let us recall the basic facts on Weyl functions.

Definition (8):

Let A be a densely defined closed symmetric operator and $\Pi=\{\mathcal{H},\Gamma_0,\Gamma_1\}$ be a boundary triple for A^*. The unique mapping $M(.)=\rho(A_0)\to[\mathcal{H}]$ defined by $\Gamma_1 f_z=M(z)\Gamma_0 f_z, f_z\in N_z=\ker(A^*-z), z\in C_+$

Is called the Weyl function corresponding to the boundary triple π .

Proposition (9):

Let A be a simple closed symmetric operator and let $\Pi=\{\mathcal{H},\Gamma_0,\Gamma_1\}$ be a boundary triple for A^* with Weyl function $M(\lambda)$. Suppose that is self-adjoint linear relation in $\mathcal{H}$ and $\lambda\in\rho(A_0)$ then

(i) $\delta(A_0)=\operatorname{supp}(M)$

(ii) $\lambda\in\rho(A_\theta)$ if and only if $\theta\in\rho(\theta-M(\lambda))$

(iii) $\lambda\in\delta_T(A_\theta)$ if and only if $O\in\delta_T(\theta-M(\lambda)).T\in\{p,c\}$

We need the following simple proposition.

Proposition (10):

Let A be a closed symmetric operator and let $\Pi=\{\mathcal{H},\Gamma_0,\Gamma_1\}$ be a boundary triple for A^*

(i) If A is simple and $\Pi_1=\{\mathcal{H}_1,\Gamma_0^1,\Gamma_1^1\}$ is another boundary triple for A^* such that $\ker(\Gamma_0)=\ker(\Gamma_1^1)$, then the Weyl functions $M(.)$ and $M_1(.)$ of Π and Π_1, respectively are related by $M_1(z)=k^*M(z)k+D,\quad z\in C_+\cup C_-$. Where $D=D^*\in[\mathcal{H}]$ and $k\in[\mathcal{H}_1,\mathcal{H}]$ is boundedly invertible.

(ii) If $\theta=G(B),B=B^*\in\mathcal{H}$, then the Weyl function $M_B(.)$ corresponding to the boundary triple $\Pi_B=\{\mathcal{H},\Gamma_0^B,\Gamma_1^B\}=\{\mathcal{H},B\Gamma_0-\Gamma_1,\Gamma_0\}$ is given by

$$M_\beta(z)=(B-M(z))^{-1},z\in\mathbb{C}_+\cup\mathbb{C}_-\quad.$$

Definition (11):

Let A be a densely defined closed symmetric operator and let $\Pi=\{\mathcal{H},\Gamma_0,\Gamma_1\}$ be a boundary triple for A^*. The mapping $\rho(A_0)\ni z\to\gamma(z)\in[\mathcal{H}\,N_z]$

$$\gamma(z)=(\Gamma_0|N_z)^{-1}:\mathcal{H}\to N_z,z\in\rho(A_0)$$

is called the –filed of the boundary triple Π . One can easily have

$$\gamma(z)=(A_0-z_0)(A_0-z_0)^{-1}\gamma(z_0),z,z_0\in\rho(A_0) \tag{10}$$

The γ-field and the Weyl function $M(.)$ are related by

$$M(z)-M(z_0)^*=(z-\bar{z}_0)\gamma(z_0)^*\gamma(z)$$

Lemma (12):

Let A be a simple densely defined closed symmetric operator on a separable Hilbert space $\mathscr{h}$ with equal deficiency indies. Further let $\Pi=\{\mathcal{H},\Gamma_0,\Gamma_1\}$ be a boundary triple for A^* with Weyl function $M(.)$. If $E_{A_0}(.)$ is the orthogonal spectral measure of A_0 define on $\mathscr{h}$ and $E_M(.)$ the associated minimal orthogonal spectral dilation of $\sum_M^0(.)$ defined on such that $E_{A_0}(\delta)=W^*E_M(\delta)W$ for any Borel set $\delta\in B(R)$.

Proof:

By (10) one obtains

$$S(M(x+iy)h,h)=y(\gamma(x+iy)h,(x+iy)h)h\in\mathcal{H} \tag{11}$$

To show this we have

$$Sm(M(z)h,h)=\frac{(M(z)h,h)-(M(z)h,h)^*}{2i}$$

Where

$$z=x+iy=|h|\left[(M(z),1)-(M(z),1)\right]/2i$$
$$=|h|\left[(z-\bar{z}_0)\gamma(z_0)^*\gamma(z)+M(z_0)-(z-\bar{z}_0)^*\gamma(z)^*-M(z_0)^*\right]/2i$$

Multiply and divided by $(z-\tilde{z}_0)\gamma(z_0)^*$

$$=\frac{|h|}{2i}\left[\frac{(z-\tilde{z}_0)\gamma(z_0)^*\gamma(z)}{(z-\tilde{z}_0)\gamma(z_0)^*}-\frac{(z-\tilde{z}_0)\gamma(z_0)\gamma(z_0)^*}{(z-\tilde{z}_0)\gamma(z_0)^*}\right]$$

$$=\frac{|h|}{2i}\left[\gamma(z)-\gamma^*(z)\right]=\frac{|h|}{2i}\left[\gamma(z)-\gamma^*(z)\right]$$

$$=\frac{|h|}{2i}\left[\gamma(z)-(\tilde{z}_0)\gamma(\tilde{z}_0)^*\gamma(z)\right]$$

$$=|h|\left[\frac{\gamma(z)-\gamma^*(z)}{2}(z-\tilde{z}_0^*)\gamma^*(z_0)\right]$$

$$=\frac{|h|}{2i}\left[(z-\tilde{z}_0)\gamma^*(\tilde{z}_0)\gamma(z)-\gamma^*(z)\gamma^*(z_0)(z-\tilde{z}_0)\right]$$

$$=\frac{|h|}{2i}\gamma^*(z_0)\left[(z-\tilde{z}_0)\gamma(z)-(z-\tilde{z}_0)\gamma^*(z)\right]$$

Where $\gamma^*(z_0)/i\,2=y\ =|h|y\left[\gamma(z),\gamma(z)\right]=y\left(\gamma(z)h,\gamma(z)h\right)$

Since $z=x+iy$, we get

$$Sm\left(M(x+iy)h,h\right)=y\left(\gamma(x+iy)h,\gamma(x+iy)h\right)$$

Which is the prove of (11). Further, it follows from (10) that

$$\gamma(x+iy)=\left[I+(x+i(y-1))(A_0-x-iy)^{-1}\right]\gamma(i) \tag{12}$$

To prove (12) we use (10)

$$\gamma(z)=(A_0-z)(A_0-z)^{-1}\gamma(z_0)$$

$$\gamma(z)=A_0(A_0-z)^{-1}\gamma z_0-z_0(A_0-z)^{-1}\gamma(z_0)$$

$$=A_0\frac{1}{A_0}\left(I-\frac{z}{A_0}\right)^{-1}\gamma(z_0)-z_0(A_0-z)^{-1}\gamma(z_0)$$

$$=\left[\left(I-ZA_0^{-1}\right)^{-1}-Z_0(A_0-z)^{-1}\right]\gamma(z_0)$$

$$=\left[\left(I+\sum_{n=1}^{\infty}Z^n\left\|A_0^{-1}\right\|^n-Z_0(A_0-Z)^{-1}\right)\right]\gamma(Z_0)$$

$$=\left[I+\sum_{n=1}^{\infty}Z^{n+1}\left\|A_0^{n+1}\right\|-Z_0\left(A_0-Z\right)^{-1}\right]\gamma\left(Z_0\right)$$

Since $A_0=A^*$ is self adjoint spectrum and $\left\|A_0^{n+1}\right\|=1$, so

$$\gamma(z)=\left[I+\sum_{n=0}^{\infty}Z^{n+1}-Z_0\left(A_0-Z\right)^{-1}\right]\gamma\left(z_0\right)$$

$$=\left[I+\sum_{n=0}^{\infty}Z^{n}.Z-Z_0\left(A_0-Z\right)^{-1}\right]\gamma\left(z_0\right)$$

But $\sum_{n=0}^{\infty}z^n=\left(A_0-z\right)^{-1}$

Hence $\gamma(z)=\left[I+Z\left(A_0-z\right)^{-1}-Z_0\left(A_0\left(A_0-Z\right)^{-1}\right)\right]\gamma\left(Z_0\right)$

$$=\left[I+\left(z-z_0\right)\left(A_0-Z\right)^{-1}\right]\gamma\left(Z_0\right)$$

Let $x=0,y=1\Rightarrow z_0=0+i$

Therefore $\gamma(z)=\left[I+(z-i)\left(A_0-z\right)^{-1}\right]\gamma(i)$

Since $z=x+iy$

$$\gamma(x+iy)=\left[I+(x+iy-i)\left(A_0-(x+iy)\right)^{-1}\right]\gamma(i)$$

$$=\left[I+\left(x+i\left(y-1\right)\right)\left(A_0-x-iy\right)^{-1}\right]\gamma(i)$$

Which is the proof of (12). Inserting (12) into (11) one gets

$$Sm\left(M\left(x+iy\right)h,h\right)=y\int_{-\infty}^{+\infty}\frac{1+t^2}{(t-x)^2+y^2}d\left(E_{A_0}(t)\gamma(i)h,\gamma(i)h\right),h\in\mathcal{H}$$

On the other hand we obtain that $d\left(\sum_M(t)h,h\right)=\left(1+t^2\right)d\left(E_{A_0}(t)\gamma(i)h,\gamma(i)h\right)$, inserting in the above representation we get

Journal of Statistics and Actuarial

Research
ISSN 2518-881X
Vol 6, Issue 1, No.2, pp 20 - 42, 2022

$$Sm\left(M\left(x+iy\right)h,h\right)=\int_{-\infty}^{+\infty}\frac{d\left(\sum_M(t)h,h\right)}{\left(t-x\right)^2+y^2},\ h\in\mathcal{H}$$

Applying the stieltjes inversion formula (8) we find

$$\left(\sum_M\left(\left(a,b\right)\right)h,h\right)=\int_{(a,b)}\left(1+t^2\right)d\left(E_{A_0}\left(t\right)\gamma\left(i\right)h,h\right),h\in\mathcal{H}$$

Which yields

$$\sum_M^0\left(\left(a,b\right)\right)=\gamma\left(i\right)^*E_{A_0}\left(\left(a,b\right)\right)\gamma\left(i\right) \tag{13}$$

for any bounded open interval $(a,b)\subseteq R$. Since A is simple it follows from (12) that

$$\left\{\left(A_0-\lambda\right)^{-1}\tan\left(\gamma\left(i\right)\right):\lambda\in C_+\cup C_-\right\}=h \tag{14}$$

By (13) and (14), $E_{A_0}(.)$ is a minimal orthogonal dilation of $\sum_M^0(.)$. By proposition (5-1-2) we find that the spectral measure $E_{A_0}(.)$ and $E_M(.)$ are unitarily equivalent.

Definition (13):

Let $\Pi=\{\mathcal{H},\Gamma_0,\Gamma_1\}$ be a boundary triple for A^* with corresponding Weyl function $M(.)$. We will call $\sum_M^0(.)$ the bounded non-orthogonal spectral measure of the extension $A_0=\left(A^*\left|\ker\left(\Gamma_0\right)\right.\right)$.

Corollary (14):

Let A be a simple densely defined closed symmetric operator in a separable Hilbert space $\mathcal{H}$ with equal deficiency indices. Further, let $\Pi=\{\mathcal{H},\Gamma_0,\Gamma_1\}$ be a boundary triple for A^* and $M(.)$ the corresponding Weyl function, then

$$\delta\left(A_0\right)=\text{supp}\left(M\right)=\text{supp}\left(\sum_M\right),\delta_T\left(A_0\right)=\text{supp}\left(\sum_M^\tau\right).$$ Where $\tau\in\{ac,s,sc,pp\}$.

Remark (15):

$M_B(.)$ of the form $M_B(z)=(B-M(z))^{-1}=(B-m(z).I_{\mathcal{H}})^{-1}$ is the Weyl function of the generalized boundary triple Π_B. Being a Wyle function. $M_B(.)$ admits the representation

$$M_B(z)=C_0+\int_{-\infty}^{+\infty}\left(\frac{1}{t-z}-\frac{t}{1+t^2}\right)d\sum_B(t),\quad z\in C_+\cup C_- \qquad (15)$$

Where $\sum_B(.)=\sum_{MB}(.)$ is the (unbounded) non-orthogonal spectral measure of $M_B(.)$. In accordance with the Stieltjes inversion formula (8) the spectral measure can be re-obtained by

$$\sum_B(a,b)=s-\lim_{\delta\to 0}s-\lim_{\epsilon\to 0}\frac{1}{2\pi i}\int_{a+\delta}^{b-\delta}\left(M_B(x+i\varepsilon)-M_B(x-i\varepsilon)\right)dx \qquad (16)$$

With $M(z)=M(\bar{z})^*$. We get

$$M_B(x+i\varepsilon)-M_B(x-i\varepsilon)=\int_{-\infty}^{+\infty}\left(\lambda-m(x+i\varepsilon)\right)^{-1}-\int_{-\infty}^{+\infty}\left(\lambda-m(x+i\varepsilon)\right)^{-1}dE_B(\lambda) \qquad (17)$$

Where $z=x+i\varepsilon$ and $z^{-*}=x-i\varepsilon$. The representation admits this

$$M_B(x+i\varepsilon)-M_B(x-i\varepsilon)=\int_{-\infty}^{+\infty}\left(\left(\lambda-m(x+i\varepsilon)\right)^{-1}-\left(\lambda-m(x-i\varepsilon)\right)^{-1}\right)dE_B(\lambda)$$

By taking the integration both sides of equation (16) which leads to the expression

$$\frac{1}{2\pi i}\int_{a+\delta}^{b-\delta}\left(M_B(x+i\varepsilon)-M_B(x-i\varepsilon)\right)dx$$

$$=\frac{1}{2\pi i}\int_{a+\delta}^{b-\delta}\int_{-\infty}^{+\infty}\left(\left(\lambda-m(x+i\varepsilon)\right)^{-1}-\left(\lambda-m(x-i\varepsilon)\right)^{-1}\right)dx$$

$$=\int_{-\infty}^{+\infty}\frac{1}{2\pi i}\int_{a+\delta}^{b-\delta}\left(\left(\lambda-m(x+i\varepsilon)\right)^{-1}-\left(\lambda-m(x-i\varepsilon)\right)^{-1}\right)dE_B(\lambda)$$

Put $=\frac{1}{2\pi i}\int_{a+\delta}^{b-\delta}\left(\left(\lambda-m(x+i\varepsilon)\right)^{-1}-\left(\lambda-m(x-i\varepsilon)\right)^{-1}\right)dx=k_\Delta(\lambda,\delta,t)$

We get the following

$$=\frac{1}{2\pi i}\int_{a+\delta}^{b-\delta}\left(\left(M_B\left(x+i\varepsilon\right)\right)^{-1}-\left(M_B\left(x-i\varepsilon\right)\right)^{-1}\right)dx=\int_{-\infty}^{+\infty}k_\Delta\left(\lambda,\delta,t\right)dE_B\left(\lambda\right),\varepsilon>0 \qquad (18)$$

and

$$k_\Delta\left(\lambda,\delta,\varepsilon\right)=\frac{1}{2\pi i}\int_{a+\delta}^{b-\delta}\left(\left(\lambda-m\left(x+i\varepsilon\right)\right)^{-1}-\left(\lambda-m\left(x+i\varepsilon\right)\right)^{-1}\right)dE_B\left(\lambda\right) \qquad (19)$$

$\lambda\varepsilon R,\Delta=\left(a,b\right)\subseteq R$ and $\varepsilon>0$ with $m\left(z\right)=\overline{m\left(\overline{z}\right)},z\varepsilon C_{\_}$ we denote by the family of the component intervals $\Delta_L=\left(a_L,b_L\right)$ of $O_m=R\setminus Supp\left(m\right)$.

Further the function $M\left(.\right)$ admits an analytic continuation to O_m such that

$$m\left(x\right)=C_0+\int_{-\infty}^{+\infty}\left(\frac{1}{t-x}-\frac{t}{1+t^2}\right)d_\mu\left(t\right),\ x\in O_m$$

Hence the function $m\left(.\right)$ restricted to O_m is analytic. Moreover one easily verifies that for every component interval Δ of O_m

$$m\left(x\right)<m\left(y\right),x<y,x,\quad y\varepsilon\Delta$$

Therefore for every component interval Δ of O_m the set $\Delta'=m\left(\Delta\right)$ is gain an open interval. Thus $O'_m=m\left(O_m\right)$ is also open and the union of the sets $O'=m\left(\Delta\right)$ where the union is taken over all component intervals Δ of O_m.

Lemma (16):

Let $m\left(.\right)$ be a scalar Nevalinna function. If $\Delta=\left(a,b\right)$ is contained in a component interval Δ_L of O_m then $C_\Delta\left(\delta\right)=\underset{\lambda\varepsilon R,\varepsilon\varepsilon(0,1]}{Sup}\left|k_\Delta\left(\lambda,\delta,\varepsilon\right)\right|<\infty$, for each

$$\delta\in\left(0,\frac{b-a}{2}\right) \qquad (20)$$

Proof:

we have

$$m(x+i\varepsilon)=m(x)-\varepsilon^2T_0(\varepsilon,x)+\overline{z}\varepsilon T_1(\varepsilon,x), x\varepsilon O_m \qquad (21)$$

Where

$$T_0(\varepsilon,x)=\int_{-\infty}^{+\infty}\frac{1}{y-x}.\frac{1}{(y-x)^2+\varepsilon^2}d_\mu(y) \qquad (22)$$

and

$$T_1(\varepsilon,x)=\int_{-\infty}^{+\infty}\frac{1}{(y-x)^2\varepsilon^2}d_\mu(y) \qquad (23)$$

using (21) and (22) we find constant $x_0(\delta),k_1(\delta)$ and $w_1(\delta)$ such that $|T_0(\varepsilon,x)|\leq x_0(\delta)$ and

$0<w_1(\delta)\leq T_1(t,x)\leq x_1(\delta)$,

$$x\in(a+\delta,b-\delta) \qquad (24)$$

For $\varepsilon\in[0,1]$ further we get from (20)

$$\begin{aligned}P(\lambda,x,\varepsilon)&=\frac{1}{\lambda-m(x+i\varepsilon)}-\frac{1}{\lambda-m(x)-i\varepsilon T_1(\varepsilon,x)}\\&=\frac{\lambda-m(x)-i\varepsilon T_1(\varepsilon,x)-\lambda+m(x+i\varepsilon)}{(\lambda-m(x+i\varepsilon))(\lambda-m(x)-i\varepsilon T_1(\varepsilon,x))}\end{aligned} \qquad (25)$$

From (20) we get

$P(\lambda,x,\varepsilon)=\frac{\varepsilon^2T_0(\varepsilon,x)}{(\lambda-m(x+i\varepsilon))(\lambda-m(x)-i\varepsilon T_1(\varepsilon,x))}$, $\lambda\in R, x\in O_m, \in>0$. Since both $m(x)$ and $T_0(\in,x)$ are real for $x\in O_m$ we have from (20) that $|\lambda-m(x+i\varepsilon)|\geq\varepsilon T_1(\varepsilon,x)$ and $|\lambda-m(x)-i\varepsilon T_1|\geq\varepsilon T_1(\varepsilon,x),\lambda\in R$. In view of (36) these inequalities yield

$$\left|p(\lambda,x,\varepsilon)\right| \le \left|\frac{T_0(t,x)}{T_1(t,x)^2}\right|, \lambda \in R, x \in O_m, \varepsilon > 0 \tag{26}$$

Combining (23) with (25) we obtain the estimate

$$\left|P(\lambda,x,\varepsilon)\right| \le \frac{x_0(\delta)}{w_1(\delta)^2}, \lambda \in R, x \in (a+\delta, b-\delta), \varepsilon \in (0,1] \tag{27}$$

We set

$$r_\Delta(\lambda,\delta,\varepsilon) = \frac{1}{2\pi i}\int_{a+\delta}^{b-\delta}\left(\frac{1}{\lambda - m(x) - i\varepsilon T_1(\varepsilon,x)} - \frac{1}{\lambda - m(x) + i\varepsilon T_1(\varepsilon,x)}\right)dx$$

for $\lambda \in R$ and $\varepsilon > 0$. By the representation

$$r_\Delta(\lambda,\delta,\varepsilon) = \frac{1}{2\pi i}\int_{a+\delta}^{b-\delta}\frac{\lambda - m(x) + i\varepsilon T_1(\varepsilon,x) - \lambda + m(x) + i\varepsilon T_1(\varepsilon,x)}{(\lambda - m(x) - i\varepsilon T_1(\varepsilon,x))(\lambda - m(x) + i\varepsilon T_1(\varepsilon,x))}dx$$

$$= \frac{1}{2\pi i}\int_{a+\delta}^{b-\delta}\left(\frac{2i\,\varepsilon T_1(\varepsilon,x)}{\left(\lambda - m(x)^2 + \varepsilon^2 T_1(\varepsilon,x)^2\right)}\right)dx$$

$$= \frac{1}{\pi}\int_{a+\delta}^{b-\delta}\left(\frac{\varepsilon T_1(\varepsilon,x)}{\left(\lambda - m(x)^2 + \varepsilon^2 T_1(\varepsilon,x)^2\right)}\right)dx$$

and the estimate (23) we obtain that $T_1(\varepsilon,x) = x_1(\delta)$ and $T_1(\varepsilon,x)^2 = w_1^2(\delta)$ put this in the above equation we get

$$r_\Delta(\lambda,\delta,\varepsilon) = \frac{1}{\pi}\int_{a+\delta}^{b-\delta}\frac{\varepsilon x_1(\delta)}{\left(\lambda - m(x)^2 + \varepsilon^2 w_1^2(\delta)\right)}dx, \lambda \in R, \varepsilon \in (0,1] \tag{28}$$

Form this equation

$$m(x) = C_0 + \int_{-\infty}^{+\infty}\left(\frac{1}{t-x} - \frac{t}{1+t^2}\right)d_\mu(t), x \in O_m$$

The derivation $m'(x), x \in O_m$, admits the representation

$$m'(x) = \int_{-\infty}^{+\infty} \frac{1}{(1-x)^2} d\mu(t), x \in O_m \tag{29}$$

Obviously, there exist constants $w_z(\delta)$ and $x_2(\delta)$ such that

$$0 < w_2(\delta) \le m'(x) \le x_2(\delta), x \in (a+\delta, b-\delta) \tag{30}$$

By combining the equation (27) and equation (29) where $0 < w_2(\delta) \le m'(x)$, $x \in (a+\delta, b-\delta)$ we have the following

$$r_\Delta(\lambda,\delta,\varepsilon) \le \frac{x_1(\delta)}{\pi w_2(\delta)} \int_{a+\delta}^{b-\delta} \frac{\varepsilon . m'(x)}{(\lambda - m(x))^2 + \varepsilon^2 w_1^2(\delta)} dx, \quad \lambda, R, \varepsilon \in (0,1].$$

Using the substitution $y = m(x)$ we derive that $\frac{dy}{dx} = m'(x)$ so $dx = \frac{dy}{m'(x)}$ in the equation we get

$$r_\Delta(\lambda,\delta,\varepsilon) \le \frac{x_1(\delta)}{\pi w_2(\delta)} \int_{a+\delta}^{b-\delta} \frac{\varepsilon . m'(x)}{(\lambda - m(x))^2 + \varepsilon^2 w_1^2(\delta)} \frac{dy}{m'(x)}$$

$$\le \frac{x_1(\delta)}{\pi w_2(\delta)} \int_{m(a+\delta)}^{m(b-\delta)} \frac{\varepsilon}{(\lambda - y)^2 + \varepsilon^2 w_1^2(\delta)} dy, \lambda \in R, \varepsilon \in (0,1]$$

Finally, we get

$$r_\Delta(\lambda,\delta,\varepsilon) \le \frac{x_1}{w_1 w_2}, \lambda \in R, \varepsilon \in (0,1] \tag{31}$$

Obviously we have

$$k_\Delta(\lambda,\delta,\varepsilon) = \frac{1}{2\pi i} \int_{a+\delta}^{b-\delta} (\rho)(\lambda,\delta,\varepsilon) - \overline{\rho(\lambda,\delta,\varepsilon)} dx + r_\Delta(\lambda,\delta,\varepsilon), \lambda \in R, \varepsilon > 0$$

Hence we find the estimate

$$|k_\Delta(\lambda,\delta,\varepsilon)| \le \frac{1}{\pi} \int_{a+\delta}^{b-\delta} |\rho(\lambda,\delta,\varepsilon)| dx + r_\Delta(\lambda,\delta,\varepsilon), \lambda \in R, \varepsilon > 0$$

Taking into account equation $\left|\rho(\lambda,\delta,\varepsilon)\right| \le \frac{x_0(\delta)}{w_1(\delta)^2}$ and the equation $r_\Delta(\lambda,\delta,\varepsilon) \le \frac{x_1}{w_1 w_2}$ we arrive at the estimate $\left|k_\Delta(\lambda,\delta,\varepsilon)\right| \le \frac{x_0}{\pi w_1(\delta)}(b-a) + \frac{x_1(\delta)}{w_1(\delta) w_2(\delta)}, \lambda \in R, \varepsilon \in (0,1]$. Which proves (19).

Since the function O_m is strictly monotone on each component interval Δ_i of O_m the inverse function $\varphi_i(.)$ exists there. The function $\varphi_i(.)$ is analytic and also strictly monotone, its first derivative $\varphi_i'(.)$ exists, it is analytic and non-negative.

Lemma (17):

Suppose that $m(.)$ is a scalar Nevanlinna function, let $\Delta = (a,b)$ be contained is some component interval Δ_i of $O_m = R \setminus \operatorname{supp}(m)$, then (with k_Δ defined as in (18)).

$$\lim_{\varepsilon \to +0} k_\Delta(\lambda,\delta,\varepsilon) = \theta_L(\lambda,\delta) = \begin{cases} 0 & \lambda \in R \setminus \left[m(a+\delta), m(b-\delta)\right] \\ \frac{1}{2}\varphi_L' \lambda \in \left\{m(a+\delta), m(b-\delta)\right\} \\ \varphi_L'(\lambda) & \lambda \in \left(m(a+\delta), m(b-\delta)\right) \end{cases} \quad (32)$$

For $\delta \in \left(0, (b-a)/2\right)$ and

$$\lim_{\varepsilon \to +0} \lim_{\varepsilon \to +0} k_\Delta(\lambda,\delta,\varepsilon) = \theta_L(\lambda,\delta) = \begin{cases} 0 & \lambda \in R \setminus \left(m(a), m(b)\right) \\ \varphi_L'(\lambda) & \lambda \in \left(m(a), m(b)\right) \end{cases} \quad (33)$$

Proof:

At first let us show that

$$\lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{a+\delta}^{b-\delta} \rho(\lambda, x, \varepsilon) dx = 0, \quad \lambda \in R \quad (34)$$

by (24) one immediately gets that

Journal of Statistics and Actuarial

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ISSN 2518-881X
Vol 6, Issue 1, No.2, pp 20 - 42, 2022

$$\lim_{\varepsilon\to 0}\rho(\lambda,x,\varepsilon)=\lim_{\varepsilon\to 0}\left(\frac{1}{\lambda-m(x+i\varepsilon)}-\frac{1}{\lambda-m(x)-i\varepsilon T_1(\varepsilon,x)}\right)$$
$$=\lim_{\varepsilon\to 0}\rho\left(\frac{\varepsilon^2 T_0(\varepsilon,x)}{(\lambda-m(x+i\varepsilon))(\lambda-m(x)-i\varepsilon T_1(\varepsilon,x))}\right)=0,\ \lambda\in R, x\in O_m, \varepsilon>0$$

Which implies that $\lim_{\varepsilon\to 0}\rho(\lambda,x,\varepsilon)=0$ by lemma (16). Now (33) is implied by (26) and the Lebesque dominated convergence theorem. Next we set Lebesque

$$T_3(t,x)=\int_{-\infty}^{+\infty}\frac{1}{(y-x)^2+\varepsilon^2}.\frac{1}{(y-x)^2}d\mu(y), x\in O_m, t\geq 0 \tag{35}$$

Obviously there is a constant $x_3(\delta)>0$ such that

$$0\leq\tau_3(\varepsilon,x)\leq x_3(\delta), x\in(a+\delta,b-\delta), \varepsilon\in[0,1] \tag{36}$$

Let

$$\rho_0(\lambda,x,t)=\frac{1}{\lambda-m(x)-i\varepsilon\tau_1(\varepsilon,x)}-\frac{1}{\lambda-m(x)-i\varepsilon T_1(0,x)}, \lambda\in R, x\in O_m \tag{37}$$

For $\varepsilon>0$, it follows from (20) , (35) and (37)

That

$$\rho_0(\lambda,x,\varepsilon)=\frac{-i\varepsilon^3\tau_3(\varepsilon,x)}{(\lambda-m(x)-i\varepsilon T_1(\varepsilon,x))(\lambda-m(x)-i\varepsilon T_1(0,x))} \tag{38}$$

For $\varepsilon>0$, since $\lambda\in R$ and $m(x)$ is real for $x\in O_m$ we get from (38)

$|\rho_0(\lambda,x,\varepsilon)|\leq\in\frac{\tau_3(\varepsilon,x)}{\tau_1(\varepsilon,x)T_1(0,x)}, \lambda\in R, x\in O_m, \varepsilon>0$ where

$$\tau_1(\varepsilon,x)=\lambda-m(x)-i\varepsilon\tau_1(\varepsilon,x),$$
$$\tau_1(0,x)=\lambda-m(x)-i\varepsilon\tau_1(0,x),$$

by using (23) and (36) we obtain the estimate

$$|\rho_0(\lambda,x,\varepsilon)|\leq\frac{\varepsilon\tau_3(\delta)}{w_1(\delta)^2}, \lambda\in R, x\in(a+\delta,b-\delta), \varepsilon\in(0,1]$$

Which immediately yields?

$$\lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int_{a+\delta}^{b-\delta}\rho_0\left(\lambda,x,\varepsilon\right)dx\ =0,\lambda\in R,\delta>0 \tag{39}$$

Finally, let us introduce

$$q_\Delta\left(\lambda,\delta,\varepsilon\right)=\frac{1}{2\pi i}\int_{a+\delta}^{b-\delta}\left(\frac{1}{\lambda-m\left(x\right)-i\varepsilon\tau_1\left(0,x\right)}-\frac{1}{\lambda-m\left(x\right)+i\varepsilon\tau_1\left(0,x\right)}\right)dx \tag{40}$$

For $\lambda\in R$ and $\varepsilon>0$. Using the representation

$$q_\Delta\left(\lambda,\delta,\varepsilon\right)=\frac{1}{2\pi i}\int_{a+\delta}^{b-\delta}\left(\frac{\left(\lambda-m\left(x\right)+i\varepsilon\tau_1\left(0,x\right)\right)-\left(\lambda-m\left(x\right)-i\varepsilon\tau_1\left(0,x\right)\right)}{\left(\lambda-m\left(x\right)-i\varepsilon\tau_1\left(0,x\right)\right)\left(\lambda-m\left(x\right)+i\varepsilon\tau_1\left(0,x\right)\right)}\right)dx$$

$$=\frac{1}{2\pi i}\int_{a+\delta}^{b-\delta}\left(\frac{2i\varepsilon\tau_1\left(0,x\right)}{\left(\lambda-m\left(x\right)\right)^2+\varepsilon^2\tau_1\left(0,x\right)^2}\right)dx$$

Form the equation (20) $\tau_1\left(0,x\right)=\int_{-\infty}^{+\infty}\frac{1}{\left(y-x\right)^2}d\mu$ and the equation $m'\left(x\right)=\int_{-\infty}^{+\infty}\frac{1}{\left(t-x\right)}d\mu\geq\left(y\right),x\in O_m$

. We get this relation $m'\left(x\right)=\tau_1\left(0,x\right),x\in O_m$ from the equation (20) and equation (28) we get after change of variable $y=m\left(x\right)$ that

$$q_\Delta\left(\lambda,\delta,\varepsilon\right)=\frac{1}{\pi}\int_{m(a+\delta)}^{(b-\delta)}\frac{\varepsilon m'\left(x\right)}{\left(\lambda-m\left(x\right)\right)^2+\varepsilon^2\tau_1\left(0,x\right)^2}dx$$

$$=\frac{1}{\pi}\int_{m(a+\delta)}^{(b-\delta)}\frac{\varepsilon m'\left(x\right)}{\left(\lambda-y\right)^2+\varepsilon^2\tau_1\left(0,\varphi_L\left(y\right)\right)^2}\frac{dx}{m'\left(x\right)},\lambda\in R,\varepsilon>0$$

$$=\frac{1}{\pi}\int_{m(a+\delta)}^{m(b-\delta)}\frac{\varepsilon}{\left(\lambda-y\right)^2+\varepsilon^2\tau_1\left(0,\varphi_L\left(y\right)\right)^2}dx$$

Where $x=\varphi_i\left(y\right)$

By $\tau_1\left(0,,\varphi_i\left(y\right)\right)=m'\left(\varphi_i\left(y\right)\right)=1/\varphi_i'\left(y\right),y\in\Delta_L$, we finally obtain that

Journal of Statistics and Actuarial

Research
ISSN 2518-881X
Vol 6, Issue 1, No.2, pp 20 - 42, 2022

www.iprjb.org

$$q_{\Delta}\left(\lambda,\delta,\varepsilon\right)=\frac{1}{\pi}\int_{m(a+\delta)}^{m(b-\delta)}\frac{t\varphi_i'\left(y\right)^2}{\varphi_i'\left(y\right)^2\left(\lambda-y\right)^2+\varepsilon^2}dy\,,\,y\in R,\varepsilon>0 \qquad (41)$$

Next we prove the relation

$$\lim_{\varepsilon\to 0}q_{\Delta}\left(\lambda,\delta,\varepsilon\right)=\theta_L\left(\lambda,\delta\right),\delta\in\left(0,\left(b-a\right)/2\right),\lambda\in R \qquad (42)$$

We consider only the case when $\lambda\in\left(m\left(a+\delta\right),m\left(b-\delta\right)\right)$. The other cases can be treated in a similar way.

Noting that $\varphi_i'\left(\lambda\right)>0$ choose an arbitrary $C\in\left(0,\varphi_i'\left(\lambda\right)\right)$. Since φ_i' is continuous we can choose $\eta>0$ such that $m\left(a+\delta\right)<\lambda-\eta<\lambda+\eta<m\left(b+a\right)$ and

$$0<\varphi_i'\left(\lambda\right)-C\le\varphi_i'\left(y\right)\le\varphi_i'\left(\lambda\right)+C\,,\lambda-\eta<y\le\lambda+\eta \qquad (43)$$

Let $a,b>0$. The change of variables $x=b\left(y-\lambda\right)/\varepsilon$ yields

$$\int_{\lambda-\eta}^{\lambda+\eta}\frac{a^2\varepsilon}{b^2\left(\lambda-y\right)^2+\varepsilon^2}dy=\frac{a^2}{\varepsilon}\int_{-\frac{b\eta}{\varepsilon}}^{\frac{b\eta}{\varepsilon}}\frac{1}{1+x^2}.\frac{\varepsilon}{b}dx\to\frac{\pi a^2}{b}\text{ as }\varepsilon\to 0 \qquad (44)$$

Setting $a=\varphi_i'\left(\lambda\right)-C$ and $b=\varphi_i'-C$ in (43) and using (44) we obtain

$$\pi\frac{\left(\varphi_i'\left(\lambda\right)-C\right)^2}{\varphi_i'\left(\lambda\right)+C}\le\liminf_{\varepsilon\to 0}\int_{\lambda-\eta}^{\lambda+\eta}\frac{\varepsilon\varphi_i'\left(y\right)^2}{\varphi_i'\left(y\right)^2\left(\lambda-y\right)^2+\varepsilon^2}dy \qquad (45)$$

$$\liminf_{\varepsilon\to 0}\int_{\lambda-\eta}^{\lambda+\eta}\frac{\varepsilon\varphi_i'\left(y\right)^2}{\varphi_i'\left(y\right)^2\left(\lambda-y\right)^2+\varepsilon^2}dy\le\pi\frac{\left(\varphi_i'\left(\lambda\right)-C\right)^2}{\varphi_i'\left(\lambda\right)+C}$$

Setting $G=\left(m\left(a+\delta\right),m\left(b-a\right)\right)\backslash\left(\lambda-\eta,\lambda+\eta\right)$ and applying the Lebesgue dominated convergence theorem we get

$$\lim_{\varepsilon\to 0}\int_G\frac{\varepsilon\varphi_i'\left(y\right)^2}{\varphi_i'\left(y\right)^2\left(\lambda-y\right)^2+\varepsilon^2}dy=0 \qquad (45)$$

By (44) and (45)

$$\pi\frac{\left(\varphi_i'(\lambda)-C\right)^2}{\varphi_i'(\lambda)+C}\le\liminf_{\varepsilon\to0}\int_{m(a+\delta)}^{m(b-\delta)}\frac{\varepsilon\varphi_i'(y)^2}{\varphi_i'(y)^2(\lambda-y)^2+\varepsilon^2}dy$$
$$\le\liminf_{\varepsilon\to0}\int_{m(a+\delta)}^{m(b-\delta)}\frac{\varepsilon\varphi_i'(y)^2}{\varphi_i'(y)^2(\lambda-y)^2+\varepsilon^2}dy\le\frac{\left(\varphi_i'(\lambda)+c\right)^2}{\varphi_i'(\lambda)-c} \tag{46}$$

Since (46) holds for every $C\in\left(0,\varphi_1'(\lambda)\right)$, (46) in combination with (40) imply (41) combining (18), (26), (36) and (39) we derive the representation

$$k_\Delta(\lambda,\delta,\varepsilon)=\frac{1}{2\pi i}\int_{a+\delta}^{b-\delta}\left(\rho(\lambda,x,\varepsilon)-\rho\overline{(\lambda,x,\varepsilon)}\right)+$$
$$\frac{1}{2\pi i}\int_{a+\delta}^{b-\delta}\left(\rho_0(\lambda,x,\varepsilon)-\rho_0\overline{(\lambda,x,\varepsilon)}\right)+q_\Delta(\lambda,x,\varepsilon) \tag{47}$$

Where $\lambda\in R$ and $\varepsilon>0$. Now combining the relation (33), (38) and (41) with (37) we arrive at (41). The relation (32) immediately follows from (31). Now we are ready to calculate a non-orthogonal spectral measure $\sum_B^0$ in a gap of any self-adjoint extension $A_B=A_B^*\in E_{Xt_A}$ if only A admits a boundary triple of a scalar-type Weyl function.

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